Analysis of Algorithms, Complexity

Κ08 Δομές Δεδομένων και Τεχνικές Προγραμματισμού
Κώστας Χατζηκοκολάκης
Outline

• How can we measure and compare algorithms meaningfully?
  - an algorithm will run at different speeds on different computers

• $O$ notation.

• Complexity types.
  - Worst-case vs average-case
  - Real-time vs amortized-time
Selection sort algorithm

```c
// Ταξινομεί τον πίνακα array μεγέθους size
void selection_sort(int array[], int size) {
  // Βρίσκουμε το μικρότερο στοιχείο του πίνακα, το τοποθετούμε στη θ
  // και συνεχίζουμε με τον ίδιο τρόπο στον υπόλοιπο πίνακα.
  for (int i = 0; i < size; i++) {
    // βρίσκουμε το μικρότερο στοιχείο από αυτά σε θέσεις >= i
    int min_position = i;
    for (int j = i; j < size; j++)
      if (array[j] < array[min_position])
        min_position = j;

    // swap των στοιχείων i και min_position
    int temp = array[i];
    array[i] = array[min_position];
    array[min_position] = temp;
  }
}
```
Running Time

- Array of 2000 integers
- Computers A, B, ..., E are progressively faster.
  - The algorithm runs faster on faster computers.

<table>
<thead>
<tr>
<th>Computer</th>
<th>Time (secs)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Computer A</td>
<td>51.915</td>
</tr>
<tr>
<td>Computer B</td>
<td>11.508</td>
</tr>
<tr>
<td>Computer C</td>
<td>2.382</td>
</tr>
<tr>
<td>Computer D</td>
<td>0.431</td>
</tr>
<tr>
<td>Computer E</td>
<td>0.087</td>
</tr>
</tbody>
</table>
More Measurements

• What about different programming languages?
• Or different compilers?
• Can we say whether algorithm A is better than B?
A more meaningful criterion

• Algorithms **consume resources**: e.g. time and space

• In some fashion that depends on the **size of the problem** solved
  - the bigger the size, the more resources an algorithm consumes

• We usually use $n$ to denote the size of the problem
  - the **length of a list** that is searched
  - the **number of items** in an array that is sorted
  - etc
## selection_sort running time

In msecs, on two types of computers

<table>
<thead>
<tr>
<th>Array Size</th>
<th>Home Computer</th>
<th>Desktop Computer</th>
</tr>
</thead>
<tbody>
<tr>
<td>125</td>
<td>12.5</td>
<td>2.8</td>
</tr>
<tr>
<td>250</td>
<td>49.3</td>
<td>11.0</td>
</tr>
<tr>
<td>500</td>
<td>195.8</td>
<td>43.4</td>
</tr>
<tr>
<td>1000</td>
<td>780.3</td>
<td>172.9</td>
</tr>
<tr>
<td>2000</td>
<td>3114.9</td>
<td>690.5</td>
</tr>
</tbody>
</table>
Curves of the running times

If we plot these numbers, they lie on the following two curves:

- \( f_1(n) = 0.0007772n^2 + 0.00305n + 0.001 \)
- \( f_2(n) = 0.0001724n^2 + 0.00040n + 0.100 \)
Discussion

- The curves have the **quadratic** form $f(n) = an^2 + bn + c$
  - difference: they have **different constants** $a, b, c$

- Different computer / programming language / compiler:
  - the curve that we get will be of the same form!

- The exact numbers change, but **the shape of the curve** stays the same.
Complexity classes, $O$-notation

- We say that an algorithm belongs to a **complexity class**
- A class is denoted by $O(g(n))$
  - $g(n)$ gives the running time as a function of the size $n$
  - it describes the **shape** of the running time curve
- For **selection_sort** the time complexity is $O(n^2)$
  - take the **dominant term** of the expression $an^2 + bn + c$
  - throw away the constant coefficient $a$
Why only the dominant term?

\[ f(n) = a n^2 + b n + c \]

with \( a = 0.0001724 \), \( b = 0.0004 \) and \( c = 0.1 \).

<table>
<thead>
<tr>
<th>( n )</th>
<th>( f(n) )</th>
<th>( a n^2 )</th>
<th>( n^2 ) term as % of total</th>
</tr>
</thead>
<tbody>
<tr>
<td>125</td>
<td>2.8</td>
<td>2.7</td>
<td>94.7</td>
</tr>
<tr>
<td>250</td>
<td>11.0</td>
<td>10.8</td>
<td>98.2</td>
</tr>
<tr>
<td>500</td>
<td>43.4</td>
<td>43.1</td>
<td>99.3</td>
</tr>
<tr>
<td>1000</td>
<td>172.9</td>
<td>172.4</td>
<td>99.7</td>
</tr>
<tr>
<td>2000</td>
<td>690.5</td>
<td>689.6</td>
<td>99.9</td>
</tr>
</tbody>
</table>
Why only the dominant term?

- The lesser term $bn + c$ contributes very little
  - even though $b, c$ are much larger than $a$
  - Thus we can ignore this lesser term

- Also: we ignore the constant $a$ in $an^2$
  - It can be thought of as the “time of a single step”
  - It depends on the computer / compiler / etc
  - We are only interested in the shape of the curve
## Common complexity classes

<table>
<thead>
<tr>
<th>$O$-notation</th>
<th>Adjective Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>$O(1)$</td>
<td>Constant</td>
</tr>
<tr>
<td>$O(\log n)$</td>
<td>Logarithmic</td>
</tr>
<tr>
<td>$O(n)$</td>
<td>Linear</td>
</tr>
<tr>
<td>$O(n \log n)$</td>
<td>Quasi-linear</td>
</tr>
<tr>
<td>$O(n^2)$</td>
<td>Quadratic</td>
</tr>
<tr>
<td>$O(n^3)$</td>
<td>Cubic</td>
</tr>
<tr>
<td>$O(2^n)$</td>
<td>Exponential</td>
</tr>
<tr>
<td>$O(10^n)$</td>
<td>Exponential</td>
</tr>
<tr>
<td>$O(2^{2^n})$</td>
<td>Doubly exponential</td>
</tr>
</tbody>
</table>
Sample running times for each class

Assume 1 step = 1 μsec.

<table>
<thead>
<tr>
<th>g(n)</th>
<th>n = 2</th>
<th>n = 16</th>
<th>n = 256</th>
<th>n = 1024</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1 μsec</td>
<td>1 μsec</td>
<td>1 μsec</td>
<td>1 μsec</td>
</tr>
<tr>
<td>log n</td>
<td>1 μsec</td>
<td>4 μsec</td>
<td>8 μsec</td>
<td>10 μsec</td>
</tr>
<tr>
<td>n</td>
<td>2 μsec</td>
<td>16 μsec</td>
<td>256 μsec</td>
<td>1.02 ms</td>
</tr>
<tr>
<td>n log n</td>
<td>2 μsec</td>
<td>64 μsec</td>
<td>2.05 ms</td>
<td>10.2 ms</td>
</tr>
<tr>
<td>n²</td>
<td>4 μsec</td>
<td>25.6 μsec</td>
<td>65.5 ms</td>
<td>1.05</td>
</tr>
<tr>
<td>n³</td>
<td>8 μsec</td>
<td>4.1 ms</td>
<td>16.8 ms</td>
<td>17.9 min</td>
</tr>
<tr>
<td>2ⁿ</td>
<td>4 μsec</td>
<td>65.5 ms</td>
<td>10⁶³ years</td>
<td>10²⁹⁷ years</td>
</tr>
</tbody>
</table>
The largest problem we can solve in time T

Assume 1 step = 1 $\mu$sec.

<table>
<thead>
<tr>
<th>$g(n)$</th>
<th>T = 1 min</th>
<th>T = 1hr</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
<td>$6 \times 10^7$</td>
<td>$3.6 \times 10^9$</td>
</tr>
<tr>
<td>$n \log n$</td>
<td>$2.8 \times 10^6$</td>
<td>$1.3 \times 10^8$</td>
</tr>
<tr>
<td>$n^2$</td>
<td>$7.75 \times 10^3$</td>
<td>$6.0 \times 10^4$</td>
</tr>
<tr>
<td>$n^3$</td>
<td>$3.91 \times 10^2$</td>
<td>$1.53 \times 10^3$</td>
</tr>
<tr>
<td>$2^n$</td>
<td>25</td>
<td>31</td>
</tr>
<tr>
<td>$10^n$</td>
<td>7</td>
<td>9</td>
</tr>
</tbody>
</table>
## Complexity of well-known algorithms

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Complexity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sequential searching of an array</td>
<td>$O(n)$</td>
</tr>
<tr>
<td>Binary searching of a sorted array</td>
<td>$O(\log n)$</td>
</tr>
<tr>
<td>Hashing (under certain conditions)</td>
<td>$O(1)$</td>
</tr>
<tr>
<td>Searching using binary search trees</td>
<td>$O(\log n)$</td>
</tr>
<tr>
<td>Selection sort, Insertion sort</td>
<td>$O(n^2)$</td>
</tr>
<tr>
<td>Quick sort, Heap sort, Merge sort</td>
<td>$O(n \log n)$</td>
</tr>
<tr>
<td>Multiplying two square $x$ matrices</td>
<td>$O(n^3)$</td>
</tr>
<tr>
<td>Traveling salesman, graph coloring</td>
<td>$O(2^n)$</td>
</tr>
</tbody>
</table>
**Formal definition of $O$-notation**

$f(n)$ is the function giving the **actual time** of the algorithm.

We say that $f(n)$ is $O(g(n))$ iff

- there exist two positive constants $K$ and $n_0$
- such that $|f(n)| \leq K|g(n)| \quad \forall n \geq n_0$.

We will **not focus** on the formal definition in this course.
Intuition

• An algorithm runs in time $O(g(n))$ iff it finishes in at most $g(n)$ steps.

• A “step” is anything that takes constant time
  - a basic operation, eg $a = b + 3$
  - a comparison, eg $if(a == 4)$
  - etc

• Typical way to compute this
  - find an expression $f(n)$ giving the exact number of steps (or an upper bound)
  - find $g(n)$ by removing the lesser terms and coefficients (justified by the formal definition)
Example

• An algorithm takes \( f(n) \) number of steps, where
  - \( f(n) = 3 + 6 + 9 + \cdots + 3n \)

• We will show that the algorithm runs in \( O(n^2) \) steps.

• First find a closed form for \( f(n) \):
  - \( f(n) = 3(1 + 2 + \cdots + n) = 3 \frac{n(n+1)}{2} = \frac{3}{2} n^2 + \frac{3}{2} n \)

• Throw away
  - the lesser term \( \frac{3}{2} n \)
  - and the coefficient \( \frac{3}{2} \)

• We get \( O(n^2) \)
Scale of strength for $O$-notation

To determine the dominant term and the lesser terms:

$$O(1) < O(\log n) < O(n) < O(n^2) < O(n^3) < O(2^n) < O(10^n)$$

Example:

• $O(6n^3 - 15n^2 + 3n \log n) = O(6n^3) = O(n^3)$
Ignoring bases of logarithms

• When we use $O$-notation, we can **ignore the bases of logarithms**
  - assume that all logarithms are in base 2.

• Changing base involves multiplying by a **constant coefficient**
  - ignored by then $O$-notation

• For example, $\log_{10} n = \frac{\log n}{\log 10}$. Notice now that $\frac{1}{\log 10}$ is a constant.
$O(1)$

- It is easy to see why the $O(1)$ notation is the right one for constant time.
- Constant time means that the algorithm finishes in $k$ steps.
- $O(k)$ is the same as $O(1)$, constants are ignored.
Caveat 1

• $O$-complexity talks about the behaviour for **large values** of $n$
  - this is why we ignore lesser terms!

• For small sizes a “bad” algorithm might be faster than a “good” one

• We can test the algorithms **experimentally** to choose the best one
Caveat 2

- $O(g(n))$ complexity is an upper bound
  - the algorithm finishes in at most $g(n)$ steps

- Comparing algorithms can be misleading!
  - item A costs at most 10 euros
  - item B costs at most 5000 euros
  - which one is cheaper?

- Programmers often say $O(g(n))$ but mean $\Theta(g(n))$
  - finishes in “exactly” $g(n)$ steps
  - we won't use $\Theta$ but keep this in mind
Types of complexities

- Depending on the **data**
  - Worst-case vs Average-case

- Depending on the **number of executions**
  - Real-time vs amortized-time
Worst-case vs Average-case

- Say we want to sort an array, **which values** are stored in the array?
- **Worst-case**: take the worst possible values
- **Average-case**: average wrt to all possible values
- Eg. quicksort
  - worst-case: $O(n^2)$ (when data are already sorted)
  - average-case: $O(n \log n)$
Real-time vs amortized-time

- **How many times** do we run the algorithm?
- **Real-time**: just once
  - \( n \) is the size of the problem
- **Armortized-time**: multiple times
  - take the average wrt all execution (not wrt the values!)
  - \( n \) is the number of executions
- Example: Dynamic array! (we will see it soon)
Some algorithms and their complexity

We will analyze the following algorithms

- Sequential search
- Selection sort
- Recursive selection sort
Sequential search

The steps to locate `target` depends on its position in `array`:
- if `target` is in `array[0]`, then we need only one step
- if `target` is in `array[i-1]`, then we need `i` steps
Complexity analysis

Worst case

- This is when \texttt{target} is in \texttt{array[size-1]}
- The algorithm needs $n$ steps
- So its complexity is $O(n)$
Complexity analysis

Average case

• Assume that we always search for a target that exists in array

• If target == array[i-1] then we need i steps

• Average wrt all possible positions i (all are equally likely)

\[
\text{Average} = \frac{1 + \ldots + n}{n} = \frac{n(n+1)}{2n} = \frac{n}{2} + \frac{1}{2}
\]

• Therefore the average is \(O(n)\)

  - Same if we consider targets that don't exist in array
Selection sort algorithm

// Ταξινομεί τον πίνακα array μεγέθους size

void selection_sort(int array[], int size) {
    // Βρίσκουμε το μικρότερο στοιχείο του πίνακα, το τοποθετούμε στη θ
    // και συνεχίζουμε με τον ίδιο τρόπο στον υπόλοιπο πίνακα.

    for (int i = 0; i < size; i++) {
        // βρίσκουμε το μικρότερο στοιχείο από αυτά σε θέσεις >= i
        int min_position = i;
        for (int j = i; j < size; j++)
            if (array[j] < array[min_position])
                min_position = j;

        // swap των στοιχείων i και min_position
        int temp = array[i];
        array[i] = array[min_position];
        a[min_position] = temp;
    }
}
Complexity analysis of selection_sort

- Inner `for`:
  - its body is constant: 1 step
  - \( n - i \) repetitions (\( n = \text{size}, i = \text{current value of } i \))
  - so the whole loop takes \( n - i \) steps

- Outer `for`:
  - its body takes \( n - i \) steps
    - +1 for the constant swapping part (ignored compared to \( n - i \))
  - first execution: \( n \) steps, second: \( n - 1 \) steps, etc
  - Total: \( n + \ldots + 1 = \frac{n(n+1)}{2} \) steps

- So the time complexity of the algorithm is \( O(n^2) \)
Recursive selection_sort

Auxiliary functions

```c
// Βρίσκει τη θέση του ελάχιστου στοιχείου στον πίνακα array
int find_min_position(int array[], int size) {
    int min_position = 0;

    for (int i = 1; i < size; i++)
        if (array[i] < array[min_position])
            min_position = i;

    return min_position
}

// Ανταλλάσσει τα στοιχεία a, b του πίνακα array
void swap (int array[], int a, int b) {
    int temp = array[a];
    array[a] = array[b];
    array[b] = temp;
}
```
Recursive selection_sort

Elegant recursive version of the algorithm

```c
// Ταξινομεί τον πίνακα array μεγέθους size
void selection_sort(int array[], int size) {
    // Με λιγότερα από 2 στοιχεία δεν έχουμε τίποτα να κάνουμε
    if (size < 2)
        return;

    // Τοποθετούμε το ελάχιστο στοιχείο στην αρχή
    swap(array, 0, find_min_position(array, size));

    // Ταξινομούμε τον υπόλοιπο πίνακα
    selection_sort(&array[1], size - 1);
}
```
Analysis of recursive selection_sort

• How many steps does selection_sort take?
  - Let $g(n)$ denote that number

• $g(0) = g(1) = 1$ (nothing to do)

• For $n > 1$ selection_sort calls:
  - find_min_position: $n$ steps
  - swap: 1 step (ignored compared to $n$)
  - selection_sort: $g(n - 1)$ steps

So $g(n) = \begin{cases} n + g(n - 1) & n > 1 \\ 1 & n \leq 1 \end{cases}$
Analysis of recursive selection_sort

This is a recurrence relation, we can solve it by unrolling:

\[ g(n) = n + g(n - 1) \]
\[ = n + (n - 1) + g(n - 2) \]
\[ = n + (n - 1) + (n - 2) + g(n - 3) \]
\[ \vdots \]
\[ = n + \ldots + 1 \]
\[ = \frac{n(n + 1)}{2} \]

So again we get complexity \( O(n^2) \)
ADTList using Linked Lists

What is the worst case complexity of each operation?

- list_insert_next
- list_remove_next
- list_next
- list_last
- list_find
Readings


• Robert Sedgewick. Αλγόριθμοι σε C, Κεφ. 2.