Analysis of Algorithms, Complexity

K08 Δομές Δεδομένων και Τεχνικές Προγραμματισμού

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Outline

- How can we measure and compare algorithms meaningfully?
    - an algorithm will run at different speeds on different computers
- O notation.
- Complexity types.
    - Worst-case vs average-case
    - Real-time vs amortized-time

Selection sort algorithm

```
// Ταξινομεί τον πίνακα array μεγέθους size

void selection_sort(int array[], int size) {
  // Βρίσκουμε το μικρότερο στοιχείο του πίνακα, το τοποθετούμε στη θ
  // και συνεχίζουμε με τον ίδιο τρόπο στον υπόλοιπο πίνακα.

  for (int i = 0; i < size; i++) {
      // βρίσκουμε το μικρότερο στοιχείο από αυτά σε θέσεις >= i
      int min_position = i;
      for (int j = i; j < size; j++)
          if (array[j] < array[min_position])
              min_position = j;

      // swap των στοιχείων i και min_position
      int temp = array[i];
      array[i] = array[min_position];
      a[min_position] = temp;
  }
}
```

Running Time

- Array of 2000 integers
- Computers A, B, …, E are progressively faster.
  - The algorithm runs faster on faster computers.

Computer	Time (secs)
Computer A	51.915
Computer B	11.508
Computer C	2.382
Computer D	0.431
Computer E	0.087

More Measurements

- What about **different programming languages**?
- Or **different compilers**?
- Can we say whether algorithm A is better than B?

A more meaningful criterion

- Algorithms **consume resources**: e.g. time and space
- In some fashion that depends on the **size of the problem** solved
  - the bigger the size, the more resources an algorithm consumes
- We usually use n to denote the size of the problem
  - the **length of a list** that is searched
  - the **number of items** in an array that is sorted
  - etc

`selection_sort` running time

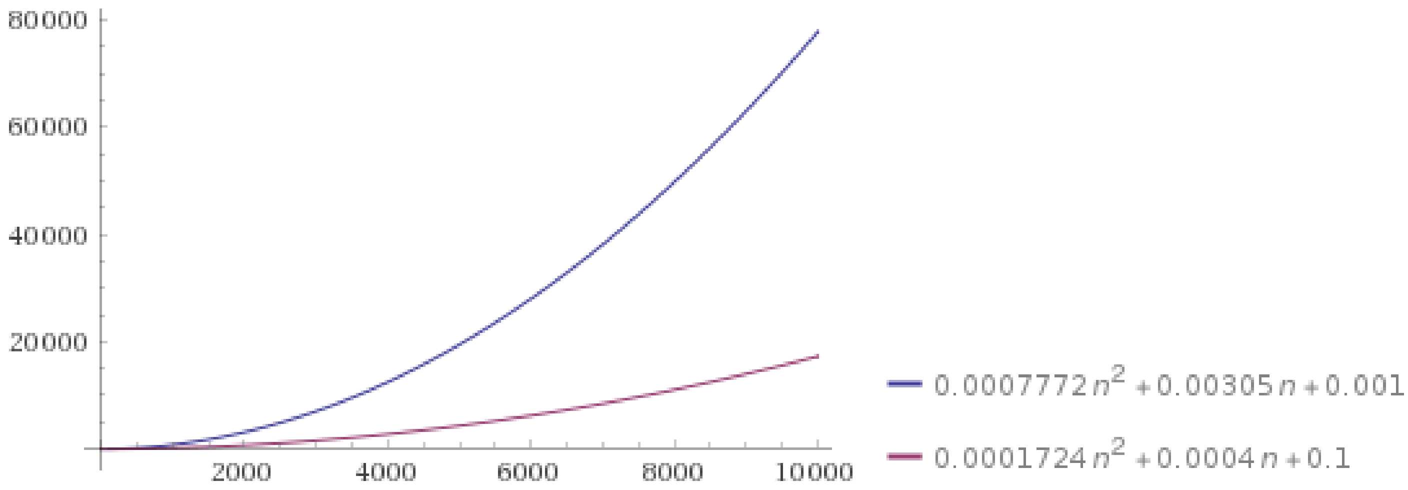
In msecs, on two types of computers

Array Size	Home Computer	Desktop Computer
125	12.5	2.8
250	49.3	11.0
500	195.8	43.4
1000	780.3	172.9
2000	3114.9	690.5

Curves of the running times

If we plot these numbers, they lie on the following two curves:

- $f_1(n) = 0.0007772n^2 + 0.00305n + 0.001$
- $f_2(n) = 0.0001724n^2 + 0.00040n + 0.100$



Discussion

- The curves have the **quadratic** form $f(n) = an^2 + bn + c$
  - difference: they have **different constants** a, b, c
- Different computer / programming language / compiler:
  - the curve that we get will be of the same form!
- The exact numbers change, but **the shape of the curve** stays the same.

Complexity classes, O-notation

- We say that an algorithm belongs to a **complexity class**
- A class is denoted by $O(g(n))$
  - $g(n)$ gives the running time as a function of the size n
  - it describes the **shape** of the running time curve
- For `selection_sort` the time complexity is $O(n^2)$
  - take the **dominant term** of the expression $an^2 + bn + c$
  - throw away the constant coefficient a

Why only the dominant term?

$f(n) = an^2 + bn + c$

with $a = 0.0001724, b = 0.0004$ and $c = 0.1$.

n	$f(n)$	an^2	n^2 **term as % of total**
125	2.8	2.7	94.7
250	11.0	10.8	98.2
500	43.4	43.1	99.3
1000	172.9	172.4	99.7
2000	690.5	689.6	99.9

Why only the dominant term?

- The lesser term $bn + c$ **contributes very little**
  - even though b, c are much larger than a
  - Thus we can **ignore this lesser term**
- Also: we **ignore the constant** a in an^2
  - It can be thought of as the “time of a single step”
  - It depends on the computer / compiler / etc
  - We are only interested in the shape of the curve

Common complexity classes

O-notation	Adjective Name
$O(1)$	Constant
$O(\log n)$	Logarithmic
$O(n)$	Linear
$O(n \log n)$	Quasi-linear
$O(n^2)$	Quadratic
$O(n^3)$	Cubic
$O(2^n)$	Exponential
$O(10^n)$	Exponential
$O(2^{2^n})$	Doubly exponential

Sample running times for each class

Assume 1 step = 1 μsec.

$g(n)$	$n = 2$	$n = 16$	$n = 256$	$n = 1024$
1	1 μsec	1 μsec	1 μsec	1 μsec
$\log_2 n$	1 μsec	4 μsec	8 μsec	10 μsec
n	2 μsec	16 μsec	256 μsec	1.02 ms
$n \log_2$	2 μsec	64 μsec	2.05 ms	10.2 ms
n^2	4 μsec	25.6 μsec	65.5 ms	1.05
n^3	8 μsec	4.1 ms	16.8 ms	17.9 min
2^n	4 μsec	65.5 ms	10^{63} years	10^{297} years

The largest problem we can solve in time T

Assume 1 step = 1 μsec.

$g(n)$	**T = 1 min**	**T = 1hr**
n	6×10^7	3.6×10^9
$n \log_2 n$	2.8×10^6	1.3×10^8
n^2	7.75×10^3	6.0×10^4
n^3	3.91×10^2	1.53×10^3
2^n	25	31
10^n	7	9

Complexity of well-known algorithms

Sequential searching of an array	$O(n)$
Binary searching of a sorted array	$O(\log n)$
Hashing (under certain conditions)	$O(1)$
Searching using binary search trees	$O(\log n)$
Selection sort, Insertion sort	$O(n^2)$
Quick sort, Heap sort, Merge sort	$O(n \log n)$
Multiplying two square x matrices	$O(n^3)$
Traveling salesman, graph coloring	$O(2^n)$

Formal definition of O-notation

$f(n)$ is the function giving the **actual time** of the algorithm.

We say that $f(n)$ is $O(g(n))$ iff

- there exist two positive constants K and n_0
- such that $|f(n)| \leq K|g(n)| \quad \forall n \geq n_0$.

We will **not focus** on the formal definition in this course.

Intuition

- An algorithm runs in time $O(g(n))$ iff it finishes in **at most** $g(n)$ **steps**.
- A "step" is anything that takes **constant time**
  - a basic operation, eg `a = b + 3`
  - a comparison, eg `if(a == 4)`
  - etc
- Typical way to compute this
  - find an expression $f(n)$ giving the exact number of steps (or an upper bound)
  - find $g(n)$ by removing the **lesser terms** and **coefficients** (justified by the formal definition)

Example

- An algorithm takes $f(n)$ number of steps, where
  - $f(n) = 3 + 6 + 9 + \cdots + 3n$
- We will show that the algorithm runs in $O(n^2)$ steps.
- First find a closed form for $f(n)$:
  - $f(n) = 3(1 + 2 + \cdots + n) = 3\frac{n(n+1)}{2} = \frac{3}{2}n^2 + \frac{3}{2}n$
- Throw away
  - the lesser term $\frac{3}{2}n$
  - and the coefficient $\frac{3}{2}$
- We get $O(n^2)$

Scale of strength for O-notation

To determine the dominant term and the lesser terms:

$$O(1) < O(\log n) < O(n) < O(n^2) < O(n^3) < O(2^n) < O(10^n)$$

Example:

- $O(6n^3 - 15n^2 + 3n \log n) = O(6n^3) = O(n^3)$

Ignoring bases of logarithms

- When we use O-notation, we can **ignore the bases of logarithms**
  - assume that all logarithms are in base 2.
- Changing base involves multiplying by a **constant coefficient**
  - ignored by then O-notation
- For example, $\log_{10} n = \frac{\log_2 n}{\log_2 10}$. Notice now that $\frac{1}{\log_2 10}$ is a constant.

$O(1)$

- It is easy to see why the $O(1)$ notation is the right one for constant time
- Constant time means that the algorithm finishes in k steps
- $O(k)$ is the same as $O(1)$, constants are ignored

Caveat 1

- O-complexity talks about the behaviour for **large values** of n
  - this is why we ignore lesser terms!
- For small sizes a “bad” algorithm might be faster than a “good” one
- We can test the algorithms **experimentally** to choose the best one

Caveat 2

- $O(g(n))$ complexity is an **upper bound**
  - the algorithm finishes in **at most** $g(n)$ steps
- Comparing algorithms can be misleading!
  - item A cost **at most 10** euros
  - item B cost **at most 5000** euros
  - which one is cheaper?
- Programmers often say $O(g(n))$ but mean $\Theta(g(n))$
  - finishes in **“exactly”** $g(n)$ steps
  - we won't use Θ but keep this in mind

Types of complexities

- Depending on the **data**
  - Worst-case vs Average-case
- Depending on the **number of executions**
  - Real-time vs amortized-time

Worst-case vs Average-case

- Say we want to sort an array, **which values** are stored in the array?
- **Worst-case**: take the worst possible values
- **Average-case**: average wrt to all possible values
- Eg. quicksort
    - worst-case: $O(n^2)$ (when data are already sorted)
    - average-case: $O(n \log n)$

Real-time vs amortized-time

- **How many times** do we run the algorithm?
- **Real-time**: just once
  - n is the size of the problem
- **Armortized-time**: multiple times
  - take the average wrt all execution (**not** wrt the **values**!)
  - n is the number of executions
- Example: Dynamic array! (we will see it soon)

Some algorithms and their complexity

We will analyze the following algorithms

- Sequential search
- Selection sort
- Recursive selection sort

Sequential search

```c
// Αναζητά τον ακέραιο target στον πίνακα target. Επιστρέφει τη θέση
// του στοιχείου αν βρεθεί, διαφορετικά -1

int sequential_search(int target, int array[], int size) {
    for (int i = 0; i < size; i++)
        if (array[i] == target)
            return i;

    return -1;
}
```

- The steps to locate `target` **depends on its position** in `array`
  - if `target` is in `array[0]`, then we need only one step
  - if `target` is in `array[i-1]`, then we need i steps

Complexity analysis

Worst case

- This is when `target` is in `array[size-1]`
- The algorithm needs n steps
- So its complexity is $O(n)$

Complexity analysis

Average case

- Assume that we always search for a `target` that **exists** in `array`
- If `target == array[i-1]` then we need i steps
- Average wrt all possible positions i (all are equally likely)

$$\text{Average} = \frac{1+\ldots+n}{n} = \frac{\frac{n(n+1)}{2}}{n} = \frac{n}{2} + \frac{1}{2}$$

- Therefore the average is $O(n)$
  - Same if we consider `target`s that don't exist in `array`

Selection sort algorithm

```
// Ταξινομεί τον πίνακα array μεγέθους size

void selection_sort(int array[], int size) {
  // Βρίσκουμε το μικρότερο στοιχείο του πίνακα, το τοποθετούμε στη θ
  // και συνεχίζουμε με τον ίδιο τρόπο στον υπόλοιπο πίνακα.

  for (int i = 0; i < size; i++) {
      // βρίσκουμε το μικρότερο στοιχείο από αυτά σε θέσεις >= i
      int min_position = i;
      for (int j = i; j < size; j++)
          if (array[j] < array[min_position])
              min_position = j;

      // swap των στοιχείων i και min_position
      int temp = array[i];
      array[i] = array[min_position];
      a[min_position] = temp;
  }
}
```

Complexity analysis of selection_sort

- Inner `for`
  - its body is constant: 1 step
  - $n - i$ repetitions (n = `size`, $i =$current value of `i`)
  - so the whole loop takes $n - i$ steps
- Outer `for`:
  - its body takes $n - i$ steps
    - +1 for the constant swapping part (ignored compared to $n - i$)
  - first execution: n steps, second: $n - 1$ steps, etc
  - Total: $n + \ldots + 1 = \frac{n(n+1)}{2}$ steps
- So the time complexity of the algorithm is $O(n^2)$

Recursive selection_sort

Auxiliary functions

```
// Βρίσκει τη θέση του ελάχιστου στοιχείου στον πίνακα array

int find_min_position(int array[], int size) {
    int min_position = 0;

    for (int i = 1; i < size; i++)
        if (array[i] < array[min_position])
            min_position = i;

    return min_position
}

// Ανταλλάσει τα στοιχεία a,b του πίνακα array

void swap (int array[], int a, int b) {
    int temp = array[a];
    array[a] = array[b];
    array[b] = temp;
}
```

Recursive selection_sort

Elegant recursive version of the algorithm

```
// Ταξινομεί τον πίνακα array μεγέθους size

void selection_sort(int array[], int size) {
    // Με λιγότερα από 2 στοιχεία δεν έχουμε τίποτα να κάνουμε
    if (size < 2)
        return;

    // Τοποθετούμε το ελάχιστο στοιχείο στην αρχή
    swap(array, 0, find_min_position(array, size));

    // Ταξινομούμε τον υπόλοιπο πίνακα
    selection_sort(&array[1], size - 1);
}
```

Analysis of recursive selection_sort

- How many steps does `selection_sort` take?
  - Let $g(n)$ denote that number
- $g(0) = g(1) = 1$ (nothing to do)
- For $n > 1$ `selection_sort` calls:
  - `find_min_position`: n steps
  - `swap`: 1 step (ignored compared to n)
  - `selection_sort`: $g(n-1)$ steps

So $g(n) = \begin{cases} n + g(n-1) & n > 1 \\ 1 & n \leq 1 \end{cases}$

Analysis of recursive selection_sort

This is a **recurrence relation**, we can solve it by **unrolling**:

$$
\begin{aligned}
g(n) &= n + g(n-1) \\
&= n + (n-1) + g(n-2) \\
&= n + (n-1) + (n-2) + g(n-3) \\
&\ldots \\
&= n + \ldots + 1 \\
&= \frac{n(n+1)}{2}
\end{aligned}
$$

So again we get complexity $O(n^2)$

ADTList using Linked Lists

What is the worst case complexity of each operation?

- `list_insert_next`
- `list_remove_next`
- `list_next`
- `list_last`
- `list_find`

Readings

- T. A. Standish. *Data Structures, Algorithms and Software Principles in C*, Chapter 6.
- Robert Sedgewick. Αλγόριθμοι σε C, Κεφ. 2.