AVL Trees

Balanced trees

• We saw that most of the algorithms in BSTs are $O(h)$
  - But $h = O(n)$ in the worst-case

• So it makes sense to keep trees “balanced”
  - Many different ways to define what “balanced” means
  - In all of them: $h = O(\log n)$

• Eg. complete are one type of balanced tree (see Heaps)
  - But it’s hard to maintain both BST and complete properties together

• AVL: a different type of balanced trees

AVL Trees

• An AVL tree is a BST with an extra property:
  For all nodes: $|\text{height(left-subtree)} - \text{height(right-subtree)}| \leq 1$

• In other words, no subtree can be much shorter/taller than the other

• Recall: height is the longest path from the root to some leaf
  - tree with only a root: height 0
  - empty tree: height -1

• Named after Russian mathematicians Adelson-Velskii and Landis

Example – AVL tree
Example – Non AVL tree

The desired property

- In an AVL tree: \( h = O(\log n) \)
  - Proving this is not hard
- \( n(h) \): minimum number of nodes of an AVL tree with height \( h \)
- We show that \( h \leq 2 \log n(h) \)
  - by induction on \( h \)
  - induction works very well on recursive structures!
- The base cases hold trivially (why?)
  - \( n(0) = 1 \)
  - \( n(1) = 2 \)

The inductive step

- Assume \( \frac{k}{2} \leq \log n(h) \) for all \( h < k \)
  - Show that it holds for an AVL tree of height \( h = k \)
- Both subtrees of the root have height at least \( h - 2 \)
  - because of the AVL property!
  - So \( n(k) \geq 2n(k - 2) \)  \( (1) \)
- Induction hypothesis for \( h = k - 2 \)
  - \( \frac{k-2}{2} \leq \log n(k - 2) \)
- From (1) we take \( \log \) on both sides and apply the ind. hypothesis
  - \( \log n(k) \geq 1 + \log n(k - 2) \geq 1 + \frac{k-2}{2} = \frac{k}{2} \)

Balance factor

A node can have one of the following “balance factors”

<table>
<thead>
<tr>
<th>Balance factor</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>-</td>
<td>Sub-trees have equal heights</td>
</tr>
<tr>
<td>/</td>
<td>Left sub-tree is 1 higher</td>
</tr>
<tr>
<td>//</td>
<td>Left sub-tree is &gt; 1 higher</td>
</tr>
<tr>
<td>\</td>
<td>Right sub-tree is 1 higher</td>
</tr>
<tr>
<td>\</td>
<td>Right sub-tree is &gt; 1 higher</td>
</tr>
</tbody>
</table>

Nodes -, /, \ are AVL.
Nodes //, \ are not AVL.
Example AVL Tree

Example AVL Tree

Example AVL Tree

Example AVL Tree
Example non-AVL Tree

Operations in an AVL Tree

- Same as those of a BST
- Except that we need to **restore** the AVL property
  - after **inserting** a node
  - or **deleting** a node
- We do this using **rotations**

Recursive AVL restore

- Restoring the AVL property is a **recursive** operation
- It happens during an insert or delete
  - Which are both recursive
  - When their recursive calls are **unwinding** towards the root
- So when we restore a node $r$, its **children** are already restored **AVL trees**

AVL restore after insert

- Assume $r$ became $\\\\\$ after an insert (the case $//\$ is symmetric)
- Let $x$ be the **root** of the **right subtree**
  - The new value was inserted under $x$ (since $r$ is $\\\\\$
- What can be the **balance factor** of $x$?
  - $\\\\\$ and $//\$ are not possible since the child $x$ is **already restored**
- Case 1: $x$ is $\\\\\$
  - A **left-rotation** on $r$ restores the property!
  - Both $r$ and $x$ become $\\\$ (easily seen in a drawing)
**AVL restore after insert**

- **Case 2:** $x$ is $/$
  - This is more tricky
  - A left-rotation on $r$ (as before) might cause $x$ to become $//$
  
  We need to do a **double right-left rotation**
  - First **right-rotation** on $x$
  - Then **left-rotation** on $r$

- The left-child $w$ of $x$ becomes the new root
  - $w$ becomes $/$
  - $r$ becomes $x$ or $/$
  - $x$ becomes $x$ or $\backslash$

---

**Case 3:** $x$ is $\backslash$

This in fact **cannot happen**!
- Assume both subtrees of $x$ have height $h$
- Then the left subtree of $r$ also must have height $(h)$
- Otherwise AVL would be violated before the insert (see the drawings)
Symmetric case

- The case when $x$ becomes $//\,$ is symmetric.
- We need to consider the BF of its left-child $x$:
  - $x$ is $/$: we do a single right rotation at $r$.
  - $x$ is $\backslash$: we do a double left-right rotation at $x$ and $r$.
  - $x$ is $\dividing$: impossible.

Insert: single right rotation at $r$

Insert: double left-right rotation at $x$ and $r$

Insert example
Inserting BRU, causes single right-rotate at ORY

Inserting DUS

Inserting ZRH

Inserting MEX
**Insert example**

Inserting ORD

**Insert example**

Inserting NRT, causes double right-left rotation at ORD and MEX

---

**AVL restore after delete**

- Assume \( r \) became \( // \) after an insert (the case \( // \) is symmetric)
- Let \( x \) be the root of the right-subtree
  - The value was deleted from the left sub-tree (since \( r \) is \( // \))
- What can be the balance factor of \( x \)?
  - \( // \) and \( // \) are not possible since the child \( x \) is already restored
- Case 1: \( x \) is \( // \)
  - A **left-rotation** on \( r \) restores the property!
  - Both \( r \) and \( x \) become \( // \) (easily seen in a drawing)

**Delete: single left-rotation at \( r \)**

Deleted node

- Height reduced
AVL restore after delete

• Case 2: $x$ is $\backslash$
  - After a **delete** this is possible!
  - A **left-rotation** on $r$ again restores the property
  - $r$ becomes $\backslash$, $x$ becomes $/$

Delete: single left-rotation at $r$

Delete: double right-left rotation at $r$

AVL restore after delete

• Case 3: $x$ is $/$
  - This is more tricky
  - A left-rotation on $r$ (as before) might cause $x$ to become $//$

  - We need to do a **double** right-left rotation
    - First **right-rotation** on $x$
    - Then **left-rotation** on $r$

  - The left-child $w$ of $x$ becomes the new root
    - $w$ becomes $\backslash$
    - $r$ becomes $\backslash$ or $/$
    - $x$ becomes $\backslash$ or $\backslash$
Deleting a, causes single left-rotate at d

Deleting m, causes double left-right rotation at d and h

Complexity of operations on AVL trees

- Search on BST is $O(h)$
  - So $O(\log n)$ for AVL, since $h \leq 2 \log n$
- Insert/delete on BST is $O(h)$
  - We add at most one rotation at each step, each rotation is $O(1)$
  - So also $O(\log n)$
- Interesting fact
  - During insert at most one rotation will be performed!
  - Because both rotations we saw decrease the height of the sub-tree
Implementation details

- We need to keep the **height** of each subtree
  - to compute the balance factors
  - If we need to save memory we can store **only** the balance factors
- Restoring after both insert and delete are similar
  - We can treat them together

Readings