AVL Trees

K08 Δομές Δεδομένων και Τεχνικές Προγραμματισμού
Κώστας Χατζηκοκολάκης
Balanced trees

- We saw that most of the algorithms in BSTs are $O(h)$
  - But $h = O(n)$ in the worst-case

- So it makes sense to keep trees “balanced”
  - Many different ways to define what “balanced” means
  - In all of them: $h = O(\log n)$

- Eg. **complete** are one type of balanced tree (see Heaps)
  - But it’s hard to maintain both BST and complete properties together

- **AVL**: a different type of balanced trees
AVL Trees

• An AVL tree is a BST with an extra property:

   For all nodes: $|\text{height(left-subtree)} - \text{height(right-subtree)}| \leq 1$

• In other words, no subtree can be much shorter/taller than the other

• Recall: \textbf{height} is the longest path from the root to some leaf

  - tree with only a root: height 0
  - empty tree: height -1

• Named after Russian mathematicians Adelson-Velskii and Landis
Example – AVL tree
Example – AVL tree
Example – AVL tree
Example – Non-AVL tree
Example – Non-AVL tree
Example – Non AVL tree
The desired property

• In an AVL tree: \( h = O(\log n) \)
  - Proving this is not hard

• \( n(h) \): minimum number of nodes of an AVL tree with height \( h \)

• We show that \( h \leq 2 \log n(h) \)
  - by induction on \( h \)
    - induction works very well on recursive structures!

• The base cases hold trivially (why?)
  - \( n(0) = 1 \)
  - \( n(1) = 2 \)
The desired property

• Inductive step
  - Assume $\frac{h}{2} \leq \log n(h)$ for all $h < k$
  - Show that it holds for an AVL tree of height $h = k$

• Both subtrees of the root have height at least $h - 2$
  - because of the AVL property!
  - So $n(k) \geq 2n(k - 2)$ \hspace{1cm} (1)

• Induction hypothesis for $h = k - 2$
  - $\frac{k-2}{2} \leq \log n(k - 2)$

• From (1) we take $\log$ on both sides and apply the ind. hypothesis
  - $\log n(k) \geq 1 + \log n(k - 2) \geq 1 + \frac{k-2}{2} = \frac{k}{2}$
# Balance factor

A node can have one of the following “balance factors”

<table>
<thead>
<tr>
<th>Balance factor</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>-</td>
<td>Sub-trees have equal heights</td>
</tr>
<tr>
<td>/</td>
<td>Left sub-tree is 1 higher</td>
</tr>
<tr>
<td>//</td>
<td>Left sub-tree is &gt; 1 higher</td>
</tr>
<tr>
<td>\</td>
<td>Right sub-tree is 1 higher</td>
</tr>
<tr>
<td>//\</td>
<td>Right sub-tree is &gt; 1 higher</td>
</tr>
</tbody>
</table>

Nodes - , / , \ are AVL.

Nodes // , \\ are not AVL.
Example AVL Tree
Example AVL Tree
Example AVL Tree
Example AVL Tree
Example AVL Tree
Example non-AVL Tree
Example non-AVL Tree
Example non-AVL Tree
Example non-AVL Tree
Operations in an AVL Tree

- Same as those of a BST
- Except that we need to **restore** the AVL property
  - after **inserting** a node
  - or **deleting** a node
- We do this using **rotations**
Recursive AVL restore

- Restoring the AVL property is a recursive operation
- It happens during an insert or delete
  - Which are both recursive
  - When their recursive calls are unwinding towards the root
- So when we restore a node $r$, its children are already restored AVL trees
AVL restore after insert

• Assume $r$ became $\backslash\backslash$ after an insert (the case $/\!/\!$ is symmetric)

• Let $x$ be the root of the right subtree
  - The new value was inserted under $x$ (since $r$ is $\backslash\backslash$)

• What can be the balance factor of $x$?
  - $\backslash\backslash$ and $/\!/\!$ are not possible since the child $x$ is already restored

• Case 1: $x$ is $\backslash$
  - A left-rotation on $r$ restores the property!
  - Both $r$ and $x$ become - (easily seen in a drawing)
Insert: single left rotation at $r$

![Diagram showing a single left rotation in a binary tree. The tree structure, including heights and new nodes, is depicted with appropriate labels for $T_1$, $T_2$, and $T_3$. The rotation process is illustrated with arrows and the heights $h+3$ and $h+2$. New nodes are marked with blue circles.](image-url)
AVL restore after insert

• Case 2: $x$ is $/$
  - This is more tricky
  - A left-rotation on $r$ (as before) might cause $x$ to become $//$

• We need to do a **double** right-left rotation
  - First **right-rotation** on $x$
  - Then **left-rotation** on $r$

• The left-child $w$ of $x$ becomes the new root
  - $w$ becomes $-$
  - $r$ becomes $-$ or $/$
  - $x$ becomes $-$ or $\backslash$
Insert: double right-left rotation at x and r

One of $T_2$ or $T_3$ has the new node and height $h$
Tree height $h+3$

Tree height $h+2$
AVL restore after insert

• Case 3: $x$ is -

• This in fact cannot happen!
  - Assume both subtrees of $x$ have height $h$
  - Then the left subtree of $r$ also must have height $(h)$
  - Otherwise AVL would be violated before the insert (see the drawings)
Symmetric case

• The case when $x$ becomes $// \text{ is symmetric}$

• We need to consider the BF of its left-child $x$
  - $x$ is $/$: we do a single right rotation at $r$
  - $x$ is $\backslash$: we do a double left-right rotation at $x$ and $r$
  - $x$ is $-$: impossible
Insert: single right rotation at $r$

New node  Tree height $h+3$

Tree height $h+2$
**Insert: double left-right rotation at x and r**

One of $T_2$ or $T_3$ has the new node and height $h$
Tree height $h+3$

Tree height $h+2$
Insert example
Insert example

Inserting BRU, causes single right-rotate at ORY
Insert example

Inserting DUS
Insert example

Inserting ZRH
Insert example

Inserting MEX
Insert example

Inserting ORD
Inserting NRT, causes double right-left rotation at ORD and MEX
AVL restore after delete

- Assume $r$ became $\backslash \backslash$ after delete (the case $//\!//\!/$ is symmetric)

- Let $x$ be the root of the right-subtree
  - The value was deleted from the left sub-tree (since $r$ is $\backslash \backslash$)

- What can be the balance factor of $x$?
  - $\backslash \backslash$ and $//\!//\!/$ are not possible since the child $x$ is already restored

- Case 1: $x$ is $\backslash$
  - A left-rotation on $r$ restores the property!
  - Both $r$ and $x$ become $-$ (easily seen in a drawing)
Delete: single left-rotation at $r$

Height reduced
AVL restore after delete

- Case 2: $x$ is -
  - After a `delete` this is possible!
  - A **left-rotation** on $r$ again restores the property
  - $r$ becomes /, $x$ becomes /
Delete: single left-rotation at $r$

Height unchanged
AVL restore after delete

• Case 3: $x$ is /
  - This is more tricky
  - A left-rotation on $r$ (as before) might cause $x$ to become //

• We need to do a double right-left rotation
  - First right-rotation on $x$
  - Then left-rotation on $r$

• The left-child $w$ of $x$ becomes the new root
  - $w$ becomes -
  - $r$ becomes - or /
  - $x$ becomes - or \\
Delete: double right-left rotation at r

Height reduced
Delete example
Delete example

Deleting a, causes single left-rotate at d
Delete example

Deleting m, causes double left-right rotation at d and h
Complexity of operations on AVL trees

• Search on BST is $O(h)$
  - So $O(\log n)$ for AVL, since $h \leq 2 \log n$

• Insert/delete on BST is $O(h)$
  - We add at most one rotation at each step, each rotation is $O(1)$
  - So also $O(\log n)$

• Interesting fact
  - During insert at most one rotation will be performed!
  - Because both rotations we saw decrease the height of the sub-tree
Implementation details

• We need to keep the **height** of each subtree
  - to compute the balance factors
  - If we need to save memory we can store **only** the balance factors

• Restoring after both insert and delete are similar
  - We can treat them together
Readings

