AVL Trees
Balanced trees

- We saw that most of the algorithms in BSTs are $O(h)$
  - But $h = O(n)$ in the worst-case
- So it makes sense to keep trees “balanced”
  - Many different ways to define what “balanced” means
  - In all of them: $h = O(\log n)$
- Eg. complete are one type of balanced tree (see Heaps)
  - But it’s hard to maintain both BST and complete properties together
- **AVL**: a different type of balanced trees
AVL Trees

• An AVL tree is a BST with an extra property:

  For **all nodes**: \(|\text{height(left-subtree)} - \text{height(right-subtree)}| \leq 1\)

• In other words, no subtree can be much shorter/taller than the other

• Recall: **height** is the longest path from the root to some leaf
  - tree with only a root: height 0
  - empty tree: height -1

• Named after Russian mathematicians Adelson-Velskii and Landis
Example – AVL tree
Example – AVL tree
Example – AVL tree
Example – Non-AVL tree
Example – Non-AVL tree
Example – Non AVL tree
The desired property

• In an AVL tree: \( h = O(\log n) \)
  - Proving this is not hard

• \( n(h) \): minimum number of nodes of an AVL tree with height \( h \)

• We show that \( h \leq 2 \log n(h) \)
  - by induction on \( h \)
  - induction works very well on recursive structures!

• The base cases hold trivially (why?)
  - \( n(0) = 1 \)
  - \( n(1) = 2 \)
The desired property

- **Inductive step**
  - Assume $\frac{h}{2} \leq \log n(h)$ for all $h < k$
  - Show that it holds for an AVL tree of height $h = k$

- **Both subtrees** of the root have height at least $h - 2$
  - because of the AVL property!
  - So $n(k) \geq 2n(k - 2)$ (1)

- Induction hypothesis for $h = k - 2$
  - $\frac{k-2}{2} \leq \log n(k - 2)$

- From (1) we take $\log$ on both sides and apply the ind. hypothesis
  - $\log n(k) \geq 1 + \log n(k - 2) \geq 1 + \frac{k-2}{2} = \frac{k}{2}$
## Balance factor

A node can have one of the following “balance factors”

<table>
<thead>
<tr>
<th>Balance factor</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>-</td>
<td>Sub-trees have equal heights</td>
</tr>
<tr>
<td>/</td>
<td>Left sub-tree is 1 higher</td>
</tr>
<tr>
<td>//</td>
<td>Left sub-tree is &gt; 1 higher</td>
</tr>
<tr>
<td>\</td>
<td>Right sub-tree is 1 higher</td>
</tr>
<tr>
<td>///</td>
<td>Right sub-tree is &gt; 1 higher</td>
</tr>
</tbody>
</table>

Nodes - , / , \ are AVL.
Nodes // , /// are not AVL.
Example AVL Tree
Example AVL Tree
Example AVL Tree
Example AVL Tree
Example AVL Tree
Example non-AVL Tree
Example non-AVL Tree
Example non-AVL Tree
Example non-AVL Tree
Operations in an AVL Tree

• Same as those of a BST

• Except that we need to **restore** the AVL property
  - after **inserting** a node
  - or **deleting** a node

• We do this using **rotations**
Recursive AVL restore

• Restoring the AVL property is a recursive operation

• It happens during an insert or delete
  - Which are both recursive
  - When their recursive calls are unwinding towards the root

• So when we restore a node $r$, its children are already restored AVL trees
AVL restore after insert

• Assume $r$ became `\` after an insert (the case `/\` is symmetric)

• Let $x$ be the root of the right subtree
  - The new value was inserted under $x$ (since $r$ is `\`)

• What can be the balance factor of $x$?
  - `\` and `/\` are not possible since the child $x$ is already restored

• Case 1: $x$ is `\`
  - A left-rotation on $r$ restores the property!
  - Both $r$ and $x$ become `-` (easily seen in a drawing)
Insert: single left rotation at $r$

- $T_1$
- $T_2$
- $T_3$

Tree height $h+3$

New node

- $T_1$
- $T_2$
- $T_3$

Tree height $h+2$
AVL restore after insert

• Case 2: $x$ is /
  - This is more tricky
  - A left-rotation on $r$ (as before) might cause $x$ to become //

• We need to do a **double** right-left rotation
  - First **right-rotation** on $x$
  - Then **left-rotation** on $r$

• The left-child $w$ of $x$ becomes the new root
  - $w$ becomes -
  - $r$ becomes - or /
  - $x$ becomes - or \
Insert: double right-left rotation at x and r

One of $T_2$ or $T_3$ has the new node and height $h$
Tree height $h+3$

Tree height $h+2$
AVL restore after insert

• Case 3: $x$ is -

• This in fact **cannot happen**!
  - Assume both subtrees of $x$ have height $h$
  - Then the left subtree of $r$ also must have height $(h)$
  - Otherwise AVL would be violated **before** the insert (see the drawings)
Symmetric case

• The case when $x$ becomes $// \text{ is symmetric}$

• We need to consider the BF of its left-child $x$
  - $x$ is $/$ : we do a single right rotation at $r$
  - $x$ is $\backslash$ : we do a double left-right rotation at $x$ and $r$
  - $x$ is $-$ : impossible
Insert: single right rotation at r

\[ T_1 \leftarrow T_2 \rightarrow T_3 \]

New node
Tree height h+3

\[ T_1 \rightarrow T_2 \leftarrow T_3 \]

Tree height h+2
Insert: double left-right rotation at x and r

One of $T_2$ or $T_3$ has the new node and height $h$
Tree height $h+3$

Tree height $h+2$
Insert example
Insert example

Inserting BRU, causes single right-rotate at ORY
Insert example

Inserting DUS
Insert example

Inserting ZRH
Insert example

Inserting MEX
Insert example

Inserting ORD
Inserting NRT, causes double right-left rotation at ORD and MEX
AVL restore after delete

• Assume \( r \) became \( \backslash\backslash \) after an insert (the case // is symmetric)

• Let \( x \) be the \textbf{root} of the \textbf{right-subtree}
  - The value was deleted from the left sub-tree (since \( r \) is \( \backslash\backslash \))

• What can be the \textbf{balance factor} of \( x \)?
  - \( \backslash\backslash \) and // are not possible since the child \( x \) is \textbf{already restored}

• Case 1: \( x \) is \( \backslash \)
  - A \textbf{left-rotation} on \( r \) restores the property!
  - Both \( r \) and \( x \) become - (easily seen in a drawing)
Delete: single left-rotation at r

Height reduced
AVL restore after delete

- Case 2: $x$ is -
  - After a **delete** this is possible!
  - A **left-rotation** on $r$ again restores the property
  - $r$ becomes \, $x$ becomes /
Delete: single left-rotation at $r$

- $r$
- $x$
- $T_1$
- $T_2$
- $T_3$

Deleted node

Height unchanged
AVL restore after delete

• Case 3: $x$ is /
  - This is more tricky
  - A left-rotation on $r$ (as before) might cause $x$ to become //

• We need to do a **double** right-left rotation
  - First **right-rotation** on $x$
  - Then **left-rotation** on $r$

• The left-child $w$ of $x$ becomes the new root
  - $w$ becomes -
  - $r$ becomes - or /
  - $x$ becomes - or \
Delete: double right-left rotation at r
Delete example
Delete example

Deleting a, causes single left-rotate at d
Delete example

Deleting m, causes double left-right rotation at d and h
Complexity of operations on AVL trees

- Search on BST is $O(h)$
  - So $O(\log n)$ for AVL, since $h \leq 2 \log n$

- Insert/delete on BST is $O(h)$
  - We add at most one rotation at each step, each rotation is $O(1)$
  - So also $O(\log n)$

- Interesting fact
  - During insert at most one rotation will be performed!
  - Because both rotations we saw decrease the height of the sub-tree
Implementation details

- We need to keep the **height** of each subtree
  - to compute the balance factors
  - If we need to save memory we can store **only** the balance factors
- Restoring after both insert and delete are similar
  - We can treat them together
Readings

