

Binary Search Trees

Κ08 Δομές Δεδομένων και Τεχνικές Προγραμματισμού

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Search

- Searching for a specific value within a large collection is fundamental
- We want this to be efficient even if we have billions of values!
- So far we have seen two basic search strategies:
 - **sequential** search: slow
 - **binary** search: fast
 - but only for **sorted** data

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Sequential search

```
// Αναζητά τον ακέραιο target στον πίνακα target. Επιστρέφει  
// τη θέση του στοιχείου αν βρεθεί, διαφορετικά -1.
```

```
int sequential_search(int target, int array[], int size) {  
    for (int i = 0; i < size; i++)  
        if (array[i] == target)  
            return i;  
  
    return -1;  
}
```

We already saw that the complexity is $O(n)$.

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Binary search

```
// Αναζητά τον ακέραιο target στον __ταξινομημένο__ πίνακα target.  
// Επιστρέφει τη θέση του στοιχείου αν βρεθεί, διαφορετικά -1.
```

```
int binary_search(int target, int array[], int size) {  
    int low = 0;  
    int high = size - 1;  
  
    while (low <= high) {  
        int middle = (low + high) / 2;  
  
        if (target == array[middle])  
            return middle; // βρέθηκε  
        else if (target > array[middle])  
            low = middle + 1; // συνεχίζουμε στο πάνω μισό  
        else  
            high = middle - 1; // συνεχίζουμε στο κάτω μισό  
    }  
  
    return -1;  
}
```

Important: the array needs to be **sorted**

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Binary search example

Searching For Result

5	50	119	210	248	270	356	425	434	519	547	604	748	874	900	941
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

At each step the search space is cut in half.

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Binary search example

```
def binarySearch(listData, value)
  low = 0
  high = len(listData) - 1
  while (low <= high)
    mid = (low + high) / 2
    if (listData[mid] == value):
      return mid
    elif (listData[mid] < value):
      low = mid + 1
    else:
      high = mid - 1
  return -1
```

Searching For Result Element found

low mid high

5	50	119	210	248	270	356	425	434	519	547	604	748	874	900	941
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

At each step the search space is cut in half.

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Complexity of binary search

- **Search space:** the elements remaining to search
 - those between **low** and **right**
- The size of the search space is **cut in half** at each step
 - After step i there are $\frac{n}{2^i}$ elements remaining
- We **stop** when $\frac{n}{2^i} < 1$
 - in other words when $n < 2^i$
 - or equivalently when $\log n < i$
- So we will do at most $\log n$ steps
 - complexity $O(\log n)$
 - **30 steps** for one **billion** elements

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Conclusions

- Binary search is fundamental for efficient search
- But we need **sorted data**
- Maintaining a sorted array **after an insert** is hard
 - complexity?
- How can we keep data sorted **and simultaneously** allow efficient inserts?

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Binary Search Trees (BST)

A **binary search tree** (δυναμικό δέντρο αναζήτησης) is a binary tree such that:

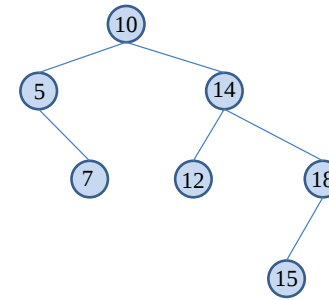
- every node is **larger** than all nodes on its **left subtree**
- every node is **smaller** than all nodes on its **right subtree**

Note

- No value can appear **twice**
(it would violate the definition)
- **Any compare** function can be used for ordering.
(with some mathematical constraints, see the piazza post)

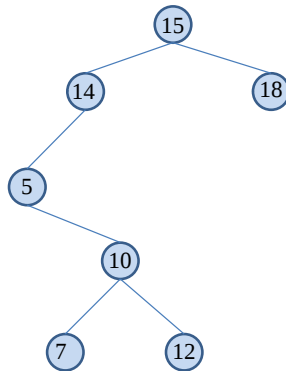
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Example



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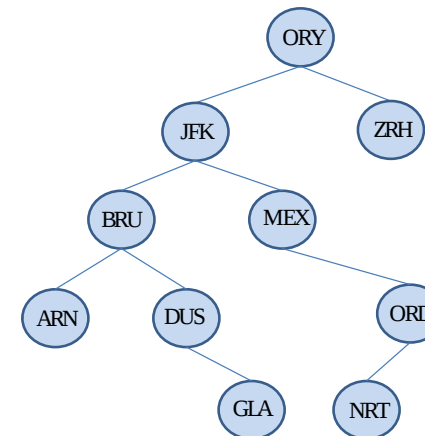
Example



A different tree with the **same values!**

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Example



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BST operations

- Container operations
 - **Insert / Remove**
- **Search** for a given value
- **Ordered** traversal
 - Find **first / last**
 - Find **next / previous**
- So we can use BSTs to implement
 - **ADTMap** (we need search)
 - **ADTSet** (we need search and ordered traversal)

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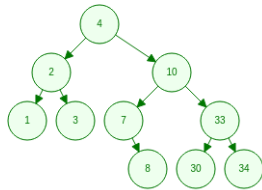
Search

We perform the following procedure **starting at the root**

- If the tree is empty
 - **target** does not exist in the tree
- If **target = current_node**
 - Found!
- If **target < current_node**
 - continue in the **left subtree**
- If **target > current_node**
 - continue in the **right subtree**

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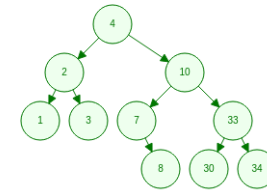
Search example



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Search example

Found:8

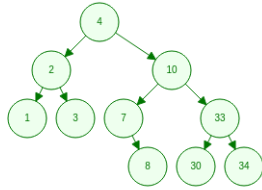


Searching for 8

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Search example

Found: 8



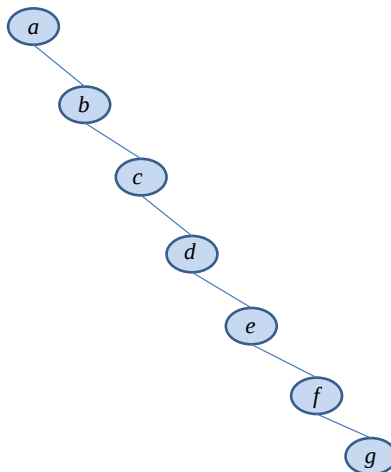
14

Complexity of search

- How many steps will we make in the worst case?
 - We will traverse a path from the root to the tree
 - h steps max (the **height** of the tree)
- But how does h relate to n ?
 - $h = O(n)$ in the worst case!
 - when the tree is essentially a degenerate "list"

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Searching in this tree is slow



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Complexity of search

- This is a very common pattern in trees
 - Many operations are $O(h)$
 - Which means worst-case $O(n)$
- Unless we manage to **keep the tree short!**
 - We already saw this in **complete** trees, in which $h \leq \log n$
- Unfortunately maintaining a complete BST is not easy (why?)
 - But there are other methods to achieve the same result
 - AVL, B-Trees, etc
 - We will talk about them later

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Inserting a new value

- Inserting a `value` is **very similar to search**
- We follow the same algorithm as if we were searching for `value`
 - If `value` is found we stop (no duplicates!)
 - If we reach an **empty subtree** insert `value` **there**

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Insert example

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Insert example



Inserting e

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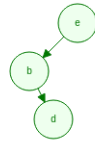
Insert example



Inserting b

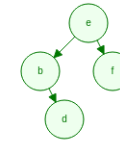
19

Insert example



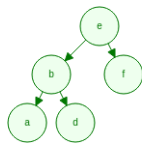
Inserting d

Insert example



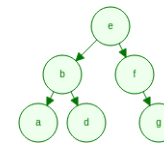
Inserting f

Insert example



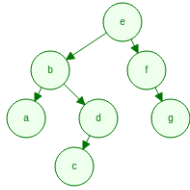
Inserting a

Insert example



Inserting g

Insert example



Inserting c

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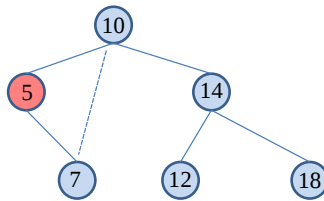
Complexity of insert

- Same as **search**
- $O(h)$
 - So $O(n)$ unless the tree is short

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Deleting a value

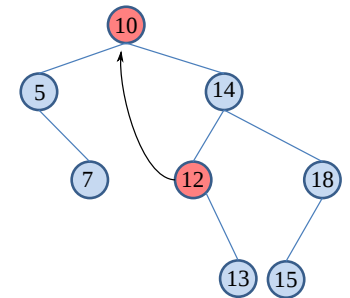
- We might want to delete **any node** in a BST
- Easy case: **node** has **as most 1 child**
- Connect the child directly to **node's parent**
- BST property is preserved (why?)



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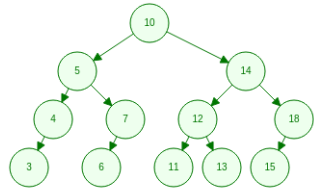
Deleting a value

- Hard case: **node** has **two children** (eg. 10)
- Find the **next** node in the order (eg. 12)
 - **left-most** node in the right sub-tree!
(or equivalently the **previous** node)
- We can replace **node's** value with **next's**
 - this preserves the BST property (why?)
- And then delete **next**
 - This has to be an **easy** case (why?)



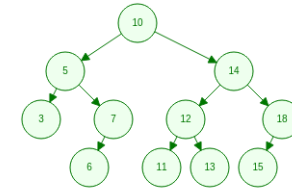
22

Delete example



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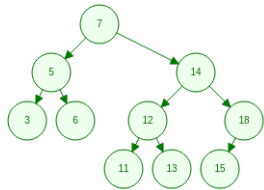
Delete example



Delete 4 (easy).

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Delete example



Delete 10 (hard). Replace with 7 and it becomes easy.

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Complexity of delete

- Finding the node to delete is $O(h)$
- Finding the next / previous is also $O(h)$

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Ordered traversal: first/last

- How to find the **first** node?
 - simply follow left children
 - $O(h)$
 - same for **last**

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Ordered traversal: next

- How to find the **next** of a given **node**?
 - Easy case: the node has a right child
 - find the left-most node of the right subtree
 - we used this for **delete!**
- Hard case: no right-child, we need to go up!

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Ordered traversal: next

General algorithm for any node.

Perform the following procedure **starting at the root**

```
// Ψευδοκώδικας, current_node είναι η ρίζα του τρέχοντος υποδέντρου,
// node είναι ο κόμβος του οποίου τον επόμενο ψάχνουμε.

find_next(current_node, node) {
    if (node == current_node) {
        // 0 target είναι η ρίζα του υποδέντρου, ο επόμενος είναι ο μ
        // του δεξιού υποδέντρου (αν είναι κενό τότε δεν υπάρχει επόμ
        return node_find_min(right_child); // NULL αν δεν υπάρχε

    } else if (node > current_node) {
        // 0 target είναι στο αριστερό υποδέντρο,
        // οπότε και ο προηγούμενός του είναι εκεί.
        return node_find_next(node->right, compare, target);

    } else {
        // 0 target είναι στο αριστερό υποδέντρο, ο επόμενός του μπορ
        // επίσης εκεί, αν όχι ο επόμενός του είναι ο ίδιος ο node.
        res = node_find_next(node->left, compare, target);
        return res != NULL ? res : node;
    }
}
```

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Complexity of next

- Similar to search, traversing the tree from the root to the leaves
 - so $O(h)$
- We can do it faster by keeping more structure
- We can keep a bidirectional list of all nodes in order
 - $O(1)$ to find next, no extra complexity to update
- More advanced: keep a **link to the parent**
 - Find the next by going **up** when needed
 - Can you find the algorithm?
 - Real-time complexity is still $O(h)$ if we traverse to the root
 - But what about amortized-time?

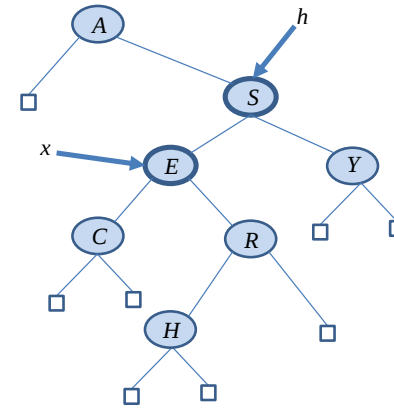
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Rotations

- **Rotation (περιστροφή)** is a fundamental operation in BSTs
 - swaps the role of a **node and one of its children**
 - while still **preserving the BST property**
- **Right rotation**
 - swap a node h and its **left child** x
 - x becomes the root of the subtree
 - the **right** child of x becomes **left** child of h
 - h becomes a **right** child of x
- **Left rotation**
 - symmetric operation with **right** child

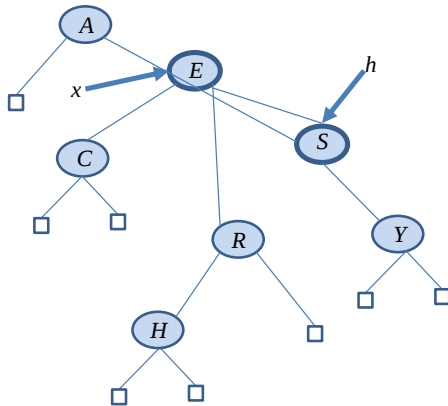
29

Example: right rotation



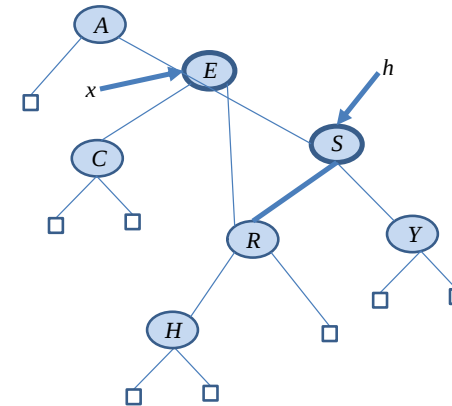
30

Example: right rotation



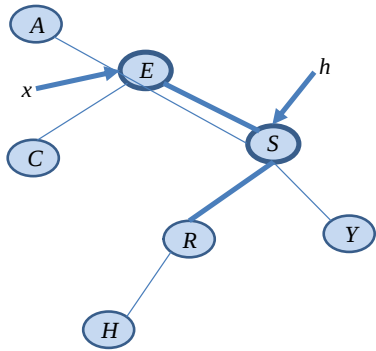
31

Example: right rotation



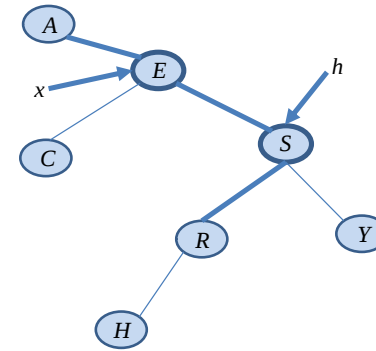
32

Example: right rotation



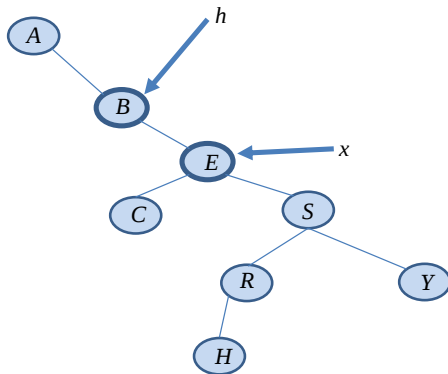
33

Example: right rotation



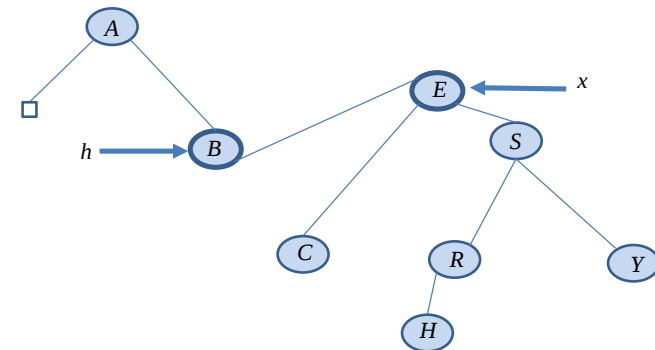
34

Example: left rotation



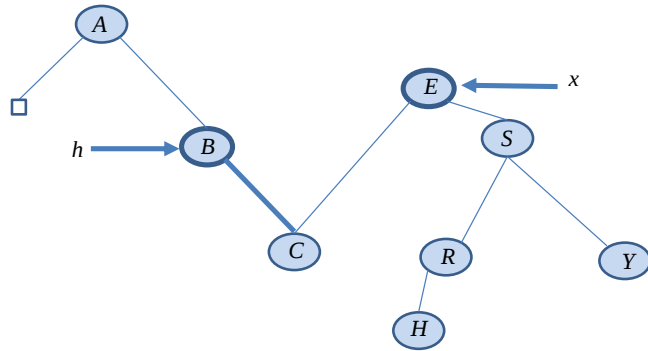
35

Example: left rotation



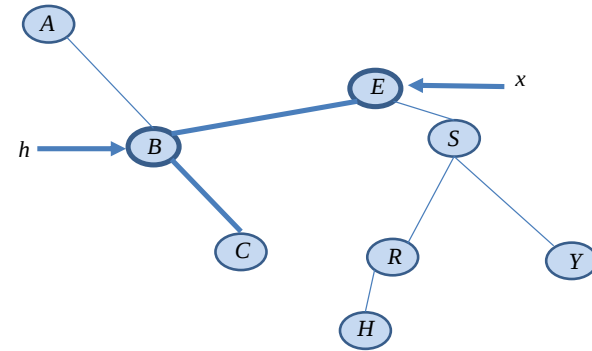
36

Example: left rotation



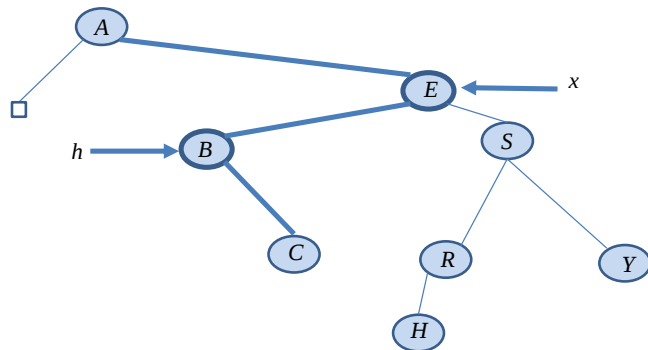
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Example: left rotation



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Example: left rotation



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Complexity of rotation

- Only changing a few pointers
- No traversal of the tree!
- So $O(1)$

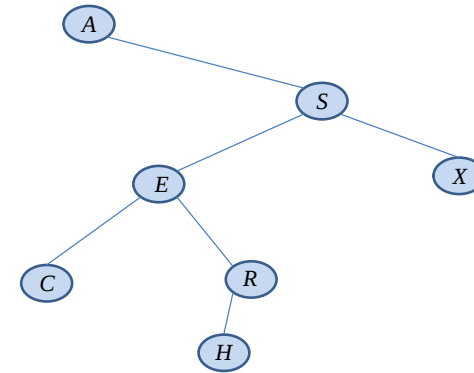
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Root insertion

- Goal
 - insert a new element
 - place it **at the root** of the tree
- Simple **recursive** algorithm using **rotations**
 1. If empty: trivial
 2. **Recursively** insert in the **left/right subtree**
 - depending on whether the value is smaller than the root or not
 - after the recursive call finishes we have a **proper BST**
 - with the value as the **root** of the left/right **subtree**
 3. **Rotate** left or right
 - the value comes at the root!

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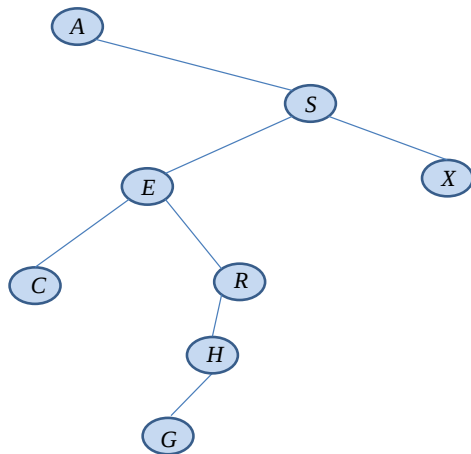
Example: root insertion



We are inserting G. The recursive algorithm is first called on the root A, then it makes **recursive calls** on the right subtree S, then on E, R, H, and finally a recursive call is made on the empty left subtree of H.

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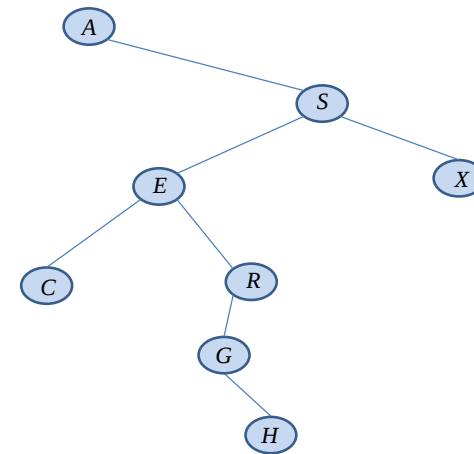
Example: root insertion



G is inserted in the empty left subtree of H.

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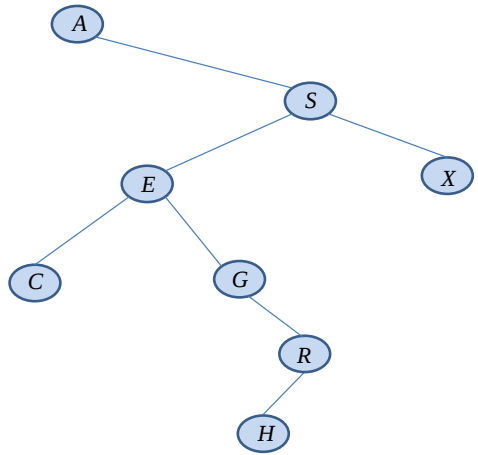
Example: root insertion



The call on H does a right rotation, G moves up.

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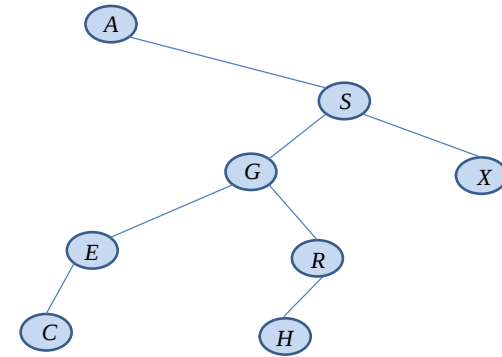
Example: root insertion



The call on R does a right rotation, G moves up.

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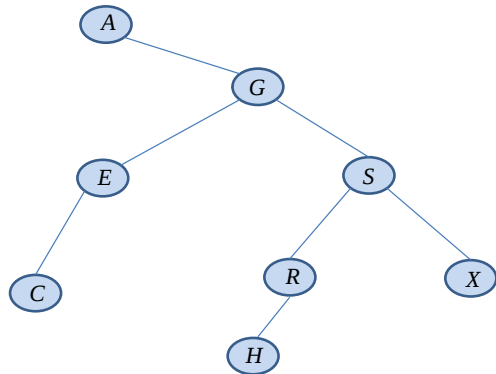
Example: root insertion



The call on E does a left rotation, G moves up.

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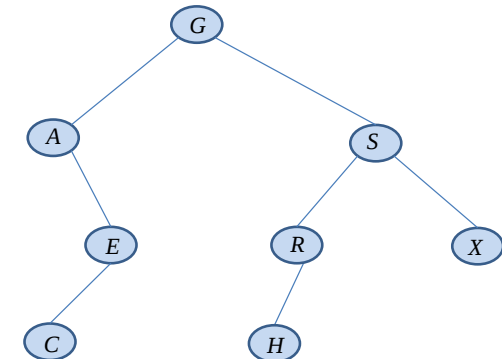
Example: root insertion



The call on R does a right rotation, G moves up.

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Example: root insertion



The call on A does a left rotation, G arrives at the root.

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Complexity of root insertion

- The algorithm is similar to a normal insert
 - traversing the tree towards the leaves: $O(h)$
- With an **extra rotation** at every step
 - which is $O(1)$
- So still $O(h)$

Readings

- T. A. Standish. *Data Structures, Algorithms and Software Principles in C*.
 - Chapter 5. Sections 5.6 and 6.5.
 - Chapter 9. Section 9.7.
- R. Sedgwick. *Αλγόριθμοι σε C*.
 - Κεφ. 12.
- M.T. Goodrich, R. Tamassia and D. Mount. *Data Structures and Algorithms in C++*. 2nd edition.
 - Section 9.3 and 10.1