A binary tree (δυαδικό δέντρο) is a set of nodes such that:

- Exactly one node is called the root
- All nodes except the root have exactly one parent
- Each node has at most two children
  - and the are ordered: called left and right

Example: a binary tree

Example: a different binary tree

Whether a child is left or right matters.
Terminology

- **path**: sequence of nodes traversing from parent to child (or vice-versa)
- **length** of a path: number of nodes - 1 (= number of "moves" it contains)
- **siblings**: children of the same parent
- **descendants**: nodes reached by travelling downwards along any path
- **ancestors**: nodes reached by travelling upwards towards the root
- **leaf / external node**: a node without children
- **internal node**: a node with children

Nodes tree can be arranged in **levels / depths**:
- The root is at **level 0**
- Its children are at **level 1**, their children are at **level 2**, etc.
- Note: node level = length of the (unique) path from the root to that node
- **height** of the tree: the largest depth of any node
- **subtree** rooted at a node: the tree consisting of that node and its descendants

Complete binary trees

A binary tree is called **complete** (πλήρες) if
- All levels except the last are "full" (have the maximum number of nodes)
- The nodes at the last level fill the level "from left to right"

Example: complete binary tree
Level order

Ordering the nodes of a tree **level-by-level** (and left-to-right in each level).

Nodes of a complete binary tree

- How many nodes does a complete binary tree have at each level?
  - At most
    - 1 at level 0.
    - 2 at level 1.
    - 4 at level 2.
    - $2^k$ at level $k$. 
Properties of binary trees

- The following hold:
  - \( h + 1 \leq n \leq 2^{h+1} - 1 \)
  - \( 1 \leq n_E \leq 2^h \)
  - \( h \leq n_I \leq 2^h - 1 \)
  - \( \log(n + 1) - 1 \leq h \leq n - 1 \)

- Where
  - \( n \): number of all nodes
  - \( n_I \): number of internal nodes
  - \( n_E \): number of external nodes (leaves)
  - \( h \): height

Properties of complete binary trees

- \( h \leq \log n \)

- Very important property, the tree cannot be too “tall”!

- Why?
  - Any level \( l < h \) contains exactly \( 2^l \) nodes
  - Level \( h \) contains at least one node
  - So \( 1 + 2 + \ldots + 2^{h-1} + 1 = 2^h \leq n \)
  - And take logarithms on both sides

How do we represent a binary tree?

Store the entries in an array at level order.

Sequential representation

- Common for complete trees
  - A lot of space is wasted for non-complete trees
    - Missing nodes will have empty slots in the array
How to find nodes

<table>
<thead>
<tr>
<th>To Find:</th>
<th>Use</th>
<th>Provided</th>
</tr>
</thead>
<tbody>
<tr>
<td>The left child of (A[i])</td>
<td>(A[2i])</td>
<td>(2i \leq n)</td>
</tr>
<tr>
<td>The right child of (A[i])</td>
<td>(A[2i + 1])</td>
<td>(2i + 1 \leq n)</td>
</tr>
<tr>
<td>The parent of (A[i])</td>
<td>(A[i/2])</td>
<td>(i &gt; 1)</td>
</tr>
<tr>
<td>The root</td>
<td>(A[1])</td>
<td>(A) is nonempty</td>
</tr>
<tr>
<td>Whether (A[i]) is a leaf</td>
<td></td>
<td>(2i &gt; n)</td>
</tr>
</tbody>
</table>

Heaps

A binary tree is called a heap (σωρός) if

- It is complete, and
- each node is greater or equal than its children

(Sometimes this is called a max-heap, we can similarly define a min-heap)

Example

Heaps and priority queues

- Heaps are a common data structure for implementing Priority Queues
- The following operations are needed
  - find max
  - insert
  - remove max
  - create with data
- We need to preserve the heap property in each operation!
Find max

- Trivial, the max is always at the root
  - remember: we always preserve the heap property
- Complexity?

Inserting a new element

The new element can only be inserted at the end
- because a heap must be a complete tree
- Now all nodes except the last satisfy the heap property
  - to restore it: apply the bubble_up algorithm on the last node

Example insertion

bubble_up(node)

- Before
  - node might be larger than its parent
  - all other nodes satisfy the heap property
- After
  - all nodes satisfy the heap property
- Algorithm
  - if node > parent
    - swap them and call bubble_up(parent)
**Example insertion**

- Inserting 15 and running bubble_up

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**Example insertion**

- Inserting 12 and running bubble_up

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**Complexity of insertion**

- We travel the tree from the last node to the root
  - on each node: 1 step (constant time)

- So we need at most $O(h)$ steps
  - $h$ is the height of the tree
  - but $h \leq \log n$ on a complete tree

- So $O(\log n)$
  - the “complete” property is crucial!

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**Removing the max element**

- We want to remove the root
  - but the heap must be a complete tree

- So swap the root with the last element
  - then remove the last element

- Now all nodes except the root satisfy the heap property
  - to restore it: apply the bubble_down algorithm on the root
Removing the max element

bubble_down(node)

- **Before**
  - node might be smaller than any of its children
  - all other nodes satisfy the heap property
- **After**
  - all nodes satisfy the heap property
- **Algorithm**
  - max_child = the largest child of node
  - If node < max_child
    - swap them and call bubble_down(max_child)

Example removal

Removing 9 and restoring the heap property

Complexity of removal

- We travel a single path from the root to a leaf
- So we need at most $O(h)$ steps
  - $h$ is the height of the tree
- Again $O(\log n)$
  - again, having a complete tree is crucial
Building a heap from initial data

- What if we want to create a heap that contains some **initial values**?
  - we call this operation **heapify**

- "Naive" implementation:
  - Create an empty heap and **insert elements one by one**

- What is the complexity of this implementation?
  - We do \(n\) inserts
  - Each insert is \(O(\log n)\) (because of bubble_up)
  - So \(O(n \log n)\) total

- Worst-case example?
  - sorted elements: each value with have to fully bubble_up to the root

Efficient heapify

- Better algorithm:
  - Visit all **internal nodes** in **reverse level order**
    - last internal node: \(\frac{n}{2}\) (parent of the last leaf \(n\))
    - first internal node: 1 (root)
  - Call **bubble_down** on each visited node

- Why does this work?
  - when we visit node, its **subtree is already a heap**
    - except from node itself (the precondition of bubble_down)
  - So **bubble_down** restores the heap property in the subtree
  - After processing the root, the whole tree is a heap

Heapify example

Visit internal nodes in inverse level order, call bubble_down.
**Complexity of heapify**

- We call `bubble_down` $\frac{n}{2}$ times
  - So $O(n \log n)$?
- But this is only an upper-bound
  - `bubble_down` is faster closer to the leaves
  - and most nodes live there!
  - we might be over-approximating the number of steps

**Efficient vs naive heapify**

- For `naive_heapify` we found $O(n \log n)$
  - maybe we are also over-approximating?
- No: in the worst-case (sorted elements) we really need $n \log n$ steps
  - try to compute the exact number of steps
- The difference:
  - `bubble_up` is faster closer to the root, but few nodes live there
  - `bubble_down` is faster closer to the leaves, and most nodes live there
- Note: in the average-case, the naive version is also $O(n)$

**Implementing ADTPriorityQueue**

// Ενα PriorityQueue είναι pointer σε αυτό το struct

struct priority_queue {
  Vector vector; // Τα δεδομένα, σε Vector για μεταβλη
  CompareFunc compare; // Η διάταξη
  DestroyFunc destroy_value; // Συνάρτηση που καταστρέφει ένα στοι
};

**Complexity of heapify**

- More careful calculation of the number of steps:
  - If node is at level $l$, `bubble_down` takes at most $h - l$ steps
  - At most $2^l$ nodes at this level, so $(h - l)2^l$ steps for level $l$
  - For the whole tree: $\sum_{l=0}^{h-1} (h - l)2^l$
  - This can be shown to be less than $2n$ (exercise if you’re curious)
- So we get worst-case $O(n)$ complexity
ADTPriorityQueue implementation

Types.

```c
// Eνα PriorityQueue είναι pointer σε αυτό το struct
struct priority_queue {
    Vector vector;     // Τα δεδομένα, σε Vector για μεταβλη
    CompareFunc compare;  // Η διάταξη
    DestroyFunc destroy_value;  // Συνάρτηση που καταστρέφει ένα στοι
};
```

Finding the max is trivial.

```c
Pointer pqueue_max(PriorityQueue pqueue) {
    return node_value(pqueue, 1);  // root
}
```

For `pqueue_insert`, the non-trivial part is `bubble_up`.

```c
static void bubble_up(PriorityQueue pqueue, int node) {
    int parent = node / 2;       // Ο πατέρας του κόμβου. Τα node ids
    // Αν ο πατέρας έχει μικρότερη τιμή από το κόμβο, swap και συνεχ
    if (pqueue->compare(node_value(pqueue, parent), node_value(pqueue, node))
        node_swap(pqueue, parent, node);
    bubble_up(pqueue, parent);
}
```

// Αποκαθιστά την ιδιότητα του σωρού.
// Πριν: όλοι οι κόμβοι ικανοποιούν την ιδιότητα του σωρού, εκτός από
//       τον node που μπορεί να είναι _μεγαλύτερος_ από τον πατέρα το
// Μετά: όλοι οι κόμβοι ικανοποιούν την ιδιότητα του σωρού.
```

```c
static void bubble_down(PriorityQueue pqueue, int node) {
    int left_child = 2 * node;
    int right_child = left_child + 1;
    int size = pqueue_size(pqueue);
    int max_child = left_child;
    if (right_child < size &
        pqueue->compare(node_value(pqueue, left_child), node_value(pqueue, right_child))
        max_child = right_child;
    // Αν ο κόμβος είναι μικρότερος από το μέγιστο παιδί, swap και συ
    if (pqueue->compare(node_value(pqueue, node), node_value(pqueue, max_child))
        node_swap(pqueue, node, max_child);
    bubble_down(pqueue, max_child);
}
```
Other possible representations

<table>
<thead>
<tr>
<th>Operation</th>
<th>Heap</th>
<th>Sorted List</th>
<th>Unsorted Vector</th>
</tr>
</thead>
<tbody>
<tr>
<td>pqueue_create (with data)</td>
<td>$O(n)$</td>
<td>$O(n \log n)$</td>
<td>$O(1)$</td>
</tr>
<tr>
<td>pqueue_remove</td>
<td>$O(\log n)$</td>
<td>$O(1)$</td>
<td>$O(n)$</td>
</tr>
<tr>
<td>pqueue_insert</td>
<td>$O(\log n)$</td>
<td>$O(n)$</td>
<td>$O(1)$</td>
</tr>
</tbody>
</table>

All of them have some advantage
- Heaps provide a great compromise between insertions and removals

Using ADTPriorityQueue for sorting
- We can easily sort data using ADTPriorityQueue
  - create a priority queue with the data
  - remove elements in sorted order
- When ADTPriorityQueue is implemented by a heap
  - this algorithm is called heapsort
  - and runs in time $O(n \log n)$

Readings
- R. Sedgewick. Αλγόριθμοι σε C. Κεφ. 5 και 9.
Proofs of given statements can be found in the following book: