Binary Trees, Heaps

Κ08 Δομές Δεδομένων και Τεχνικές Προγραμματισμού
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Binary trees

A binary tree (δυαδικό δέντρο) is a set of nodes such that:

- Exactly one node is called the root
- All nodes except the root have exactly one parent
- Each node has at most two children
  - and they are ordered: called left and right
Example: a binary tree
Example: a different binary tree

Whether a child is left or right matters.
Terminology

- **path**: sequence of nodes traversing from parent to child (or vice-versa)
- **length** of a path: number of nodes -1 (= number of “moves” it contains)
- **siblings**: children of the same parent
- **descendants**: nodes reached by travelling downwards along any path
- **ancestors**: nodes reached by travelling upwards towards the root
- **leaf / external node**: a node without children
- **internal node**: a node with children
Terminology

• Nodes tree can be arranged in levels / depths:
  - The root is at level 0
  - Its children are at level 1, their children are at level 2, etc.

• Note: node level = length of the (unique) path from the root to that node

• height of the tree: the largest depth of any node

• subtree rooted at a node: the tree consisting of that node and its descendants
A binary tree is called **complete** (πλήρες) if

- All levels except the last are **“full”** (have the maximum number of nodes)
- The nodes at the last level fill the level “from left to right”
Example: complete binary tree
Example: not complete binary tree
Example: not complete binary tree
Level order

Ordering the nodes of a tree **level-by-level** (and left-to-right in each level).
**Nodes of a complete binary tree**

- How many nodes does a complete binary tree have at each level?
  - At most
    - 1 at level 0.
    - 2 at level 1.
    - 4 at level 2.
    - ...  
    - \(2^k\) at level \(k\).
Properties of binary trees

- The following hold:
  - \( h + 1 \leq n \leq 2^{h+1} - 1 \)
  - \( 1 \leq n_E \leq 2^h \)
  - \( h \leq n_I \leq 2^h - 1 \)
  - \( \log(n + 1) - 1 \leq h \leq n - 1 \)

- Where
  - \( n \): number of all nodes
  - \( n_I \): number of internal nodes
  - \( n_E \): number of external nodes (leaves)
  - \( h \): height
Properties of complete binary trees

\[ h \leq \log n \]

- Very important property, the tree cannot be too “tall”!
- Why?
  - Any level \( l < h \) contains exactly \( 2^l \) nodes
  - Level \( h \) contains at least one node
  - So \( 1 + 2 + \ldots + 2^{h-1} + 1 = 2^h \leq n \)
  - And take logarithms on both sides
How do we represent a binary tree?

```
- H
  - D
    - B
      - A
    - F
      - C
      - E
  - K
    - J
    - L

1 2 3
4 5 6
7
8 9 10 11 12
```
Sequential representation

Store the entries in an array at level order.

<table>
<thead>
<tr>
<th>A:</th>
<th>H</th>
<th>D</th>
<th>K</th>
<th>B</th>
<th>F</th>
<th>J</th>
<th>L</th>
<th>A</th>
<th>C</th>
<th>E</th>
<th>G</th>
<th>I</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
<td>11</td>
<td>12</td>
</tr>
</tbody>
</table>

- Common for complete trees
- A lot of space is wasted for non-complete trees
  - missing nodes will have empty slots in the array
## How to find nodes

<table>
<thead>
<tr>
<th>To Find:</th>
<th>Use</th>
<th>Provided</th>
</tr>
</thead>
<tbody>
<tr>
<td>The left child of $A[i]$</td>
<td>$A[2i]$</td>
<td>$2i \leq n$</td>
</tr>
<tr>
<td>The right child of $A[i]$</td>
<td>$A[2i + 1]$</td>
<td>$2i + 1 \leq n$</td>
</tr>
<tr>
<td>The parent of $A[i]$</td>
<td>$A[i/2]$</td>
<td>$i &gt; 1$</td>
</tr>
<tr>
<td>The root</td>
<td>$A[1]$</td>
<td>$A$ is nonempty</td>
</tr>
<tr>
<td>Whether $A[i]$ is a leaf</td>
<td></td>
<td>$2i &gt; n$</td>
</tr>
</tbody>
</table>
Heaps

A binary tree is called a heap (σωρός) if

• It is **complete**, and

• each node is **greater or equal than its children**

(Sometimes this is called a **max-heap**, we can similarly define a min-heap)
Example

```
10
 /   \
9    8
 / \
7   6
 | \
1   4
   |
   3
   |
   5
   \
   2
```
Heaps and priority queues

• Heaps are a common data structure for implementing Priority Queues

• The following operations are needed
  - find max
  - insert
  - remove max
  - create with data

• We need to preserve the heap property in each operation!
Find max

• Trivial, the max is always at the root  
  - remember: we always preserve the heap property

• Complexity?
Inserting a new element

• The new element can only be inserted at the end
  - because a heap must be a complete tree

• Now all nodes except the last satisfy the heap property
  - to restore it: apply the bubble_up algorithm on the last node
Inserting a new element

bubble_up(node)

• Before
  - node might be larger than its parent
  - all other nodes satisfy the heap property

• After
  - all nodes satisfy the heap property

• Algorithm
  - if node > parent
    - swap them and call bubble_up(parent)
Example insertion
Example insertion

Inserting 15 and running bubble_up
Example insertion

Inserting 12 and running bubble_up
Complexity of insertion

- We travel the tree from the last node to the root
  - on each node: 1 step (constant time)

- So we need at most $O(h)$ steps
  - $h$ is the height of the tree
  - but $h \leq \log n$ on a complete tree

- So $O(\log n)$
  - the “complete” property is crucial!
Removing the max element

• We want to remove the root
  - but the heap must be a complete tree

• So swap the root with the last element
  - then remove the last element

• Now all nodes except the root satisfy the heap property
  - to restore it: apply the bubble_down algorithm on the root
Removing the max element

bubble_down(node)

• **Before**
  - node might be **smaller** than any of its children
  - all other nodes satisfy the heap property

• **After**
  - all nodes satisfy the heap property

• **Algorithm**
  - max_child = the **largest child** of node
  - If node < max_child
    - **swap them** and call bubble_down(max_child)
Example removal
Example removal

Removing element 9

<table>
<thead>
<tr>
<th>-INF</th>
<th>8</th>
<th>7</th>
<th>6</th>
<th>4</th>
<th>3</th>
<th>2</th>
<th>5</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
</tr>
<tr>
<td></td>
<td>9</td>
<td>10</td>
<td>11</td>
<td>12</td>
<td>13</td>
<td>14</td>
<td>15</td>
<td>16</td>
</tr>
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<td>17</td>
<td>18</td>
<td>19</td>
<td>20</td>
<td>21</td>
<td>22</td>
<td>23</td>
<td>24</td>
</tr>
<tr>
<td></td>
<td>25</td>
<td>26</td>
<td>27</td>
<td>28</td>
<td>29</td>
<td>30</td>
<td>31</td>
<td></td>
</tr>
</tbody>
</table>

Removing 9 and restoring the heap property
Complexity of removal

• We travel a single path from the root to a leaf
• So we need at most $O(h)$ steps
  - $h$ is the height of the tree
• Again $O(\log n)$
  - again, having a complete tree is crucial
Building a heap from initial data

• What if we want to create a heap that contains some initial values?
  - we call this operation heapify

• “Naive” implementation:
  - Create an empty heap and insert elements one by one

• What is the complexity of this implementation?
  - We do \( n \) inserts
  - Each insert is \( O(\log n) \) (because of bubble_up)
  - So \( O(n \log n) \) total

• Worst-case example?
  - sorted elements: each value with have to fully bubble_up to the root
Efficient heapify

• Better algorithm:
  - Visit all **internal nodes** in **reverse level order**
    - last internal node: \( \frac{n}{2} \) (parent of the last leaf \( n \))
    - first internal node: 1 (root)
  - Call **bubble_down** on each visited **node**

• Why does this work?
  - when we visit **node**, its **subtree is already a heap**
    - except from **node itself** (the precondition of **bubble_down**)
  - So **bubble_down** restores the heap property **in the subtree**
  - After processing the root, the whole tree is a heap
Heapify example

```
-\infty
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31
```
Heapify example

Visit internal nodes in inverse level order, call bubble_down.
Complexity of heapify

• We call `bubble_down` \( \frac{n}{2} \) times
  - So \( O(n \log n) \)?

• But this is only an upper-bound
  - `bubble_down` is faster closer to the leaves
  - and most nodes live there!
  - we might be over-approximating the number of steps
Complexity of heapify

• More careful calculation of the number of steps:
  - If node is at level $l$, bubble_down takes at most $h - l$ steps
  - At most $2^l$ nodes at this level, so $(h - l)2^l$ steps for level $l$
  - For the whole tree: $\sum_{l=0}^{h-1} (h - l)2^l$
  - This can be shown to be less than $2n$ (exercise if you're curious)

• So we get worst-case $O(n)$ complexity
Efficient vs naive heapify

• For naive_heapify we found $O(n \log n)$
  - maybe we are also over-approximating?
• No: in the worst-case (sorted elements) we really need $n \log n$ steps
  - try to compute the exact number of steps
• The difference:
  - bubble_up is faster closer to the root, but few nodes live there
  - bubble_down is faster closer to the leaves, and most nodes live there
• Note: in the average-case, the naive version is also $O(n)$
Implementing ADTPriorityQueue

Types

```c
// Eva PriorityQueue είναι pointer σε αυτό το struct

struct priority_queue {
    Vector vector; // Τα δεδομένα, σε Vector για μεταβλη
    CompareFunc compare; // Η διάταξη
    DestroyFunc destroy_value; // Συνάρτηση που καταστρέφει ένα στοι
};
```
ADTPriorityQueue implementation

Types.

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```
ADTPriorityQueue implementation

Finding the max is trivial.

```c
Pointer pqueue_max(PriorityQueue pqueue) {
    return node_value(pqueue, 1);  // root
}
```
**ADTPriorityQueue implementation**

For `pqueue_insert`, the non-trivial part is `bubble_up`.

```c
// Αποκαθιστά την ιδιότητα του σωρού.
// Πριν: όλοι οι κόμβοι ικανοποιούν την ιδιότητα του σωρού, εκτός από τον κόμβο που μπορεί να είναι _μεγαλύτερος_ από τον πατέρα του.
// Μετά: όλοι οι κόμβοι ικανοποιούν την ιδιότητα του σωρού.

static void bubble_up(PriorityQueue pqueue, int node) {
    // Αν φτάσαμε στη ρίζα, σταματάμε
    if (node == 1)
        return;

    int parent = node / 2;  // Ο πατέρας του κόμβου. Τα node ids
    // Αν ο πατέρας έχει μικρότερη τιμή από τον κόμβο, swap και συνεχ
    if (pqueue->compare(node_value(pqueue, parent), node_value(pqueue, node))
        node_swap(pqueue, parent, node);
    bubble_up(pqueue, parent);
}
```
ADTPriorityQueue implementation

// Πριν: όλοι οι κόμβοι ικανοποιούν την ιδιότητα του σωρού, εκτός από // node που μπορεί να είναι _μικρότερος_ από κάποιο από τα παιδ // Μετά: όλοι οι κόμβοι ικανοποιούν την ιδιότητα του σωρού.

static void bubble_down(PriorityQueue pqueue, int node) {
    // βρίσκουμε τα παιδιά του κόμβου (αν δεν υπάρχουν σταματάμε)
    int left_child = 2 * node;
    int right_child = left_child + 1;
    int size = pqueue_size(pqueue);
    if (left_child > size)
        return;

    // βρίσκουμε το μέγιστο από τα 2 παιδιά
    int max_child = left_child;
    if (right_child <= size && pqueue->compare(node_value(pqueue, lef
        max_child = right_child;

    // Αν ο κόμβος είναι μικρότερος από το μέγιστο παιδί, swap και συ
    if (pqueue->compare(node_value(pqueue, node), node_value(pqueue, node_swap(pqueue, node, max_child);
        bubble_down(pqueue, max_child);
    }
}
## Other possible representations

<table>
<thead>
<tr>
<th>Operation</th>
<th>Heap O(n)</th>
<th>Sorted List O(n log n)</th>
<th>Unsorted Vector O(1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>pqueue_create (with data)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>pqueue_remove</td>
<td>O(log n)</td>
<td>O(1)</td>
<td>O(n)</td>
</tr>
<tr>
<td>pqueue_insert</td>
<td>O(log n)</td>
<td>O(n)</td>
<td>O(1)</td>
</tr>
</tbody>
</table>

All of them have **some** advantage

- Heaps provide a great compromise between insertions and removals
Using ADTPriorityQueue for sorting

• We can easily sort data using ADTPriorityQueue
  - create a priority queue with the data
  - remove elements in sorted order

• When ADTPriorityQueue is implemented by a heap
  - this algorithm is called heapsort
  - and runs in time $O(n \log n)$
Readings


• R. Sedgewick. *Αλγόριθμοι σε C*. Κεφ. 5 και 9.

Proofs of given statements can be found in the following book: