Binary Trees, Heaps

Κ08 Δομές Δεδομένων και Τεχνικές Προγραμματισμού
Κώστας Χατζηκοκολάκης
Binary trees

A binary tree (δυαδικό δέντρο) is a set of nodes such that:

- Exactly one node is called the root
- All nodes except the root have exactly one parent
- Each node has at most two children
  - and the are ordered: called left and right
Example: a binary tree

```
R
 /   \
S     T
 / \
X   Y  U  V
 /     \
Z       W
```
Example: a different binary tree

Whether a child is left or right matters.
Terminology

- **path**: sequence of nodes traversing from parent to child (or vice-versa)
- **length** of a path: number of nodes - 1 (= number of “moves” it contains)
- **siblings**: children of the same parent
- **descendants**: nodes reached by travelling downwards along any path
- **ancestors**: nodes reached by travelling upwards towards the root
- **leaf / external node**: a node without children
- **internal node**: a node with children
Terminology

• Nodes tree can be arranged in **levels / depths:**
  - The root is at **level 0**
  - Its children are at **level 1**, their children are at **level 2**, etc.

• Note: node level = length of the (unique) path from the root to that node

• **height** of the tree: the largest depth of any node

• **subtree** rooted at a node: the tree consisting of that node and its descendants
Complete binary trees

A binary tree is called complete (πλήρες) if

• All levels except the last are “full” (have the maximum number of nodes)

• The nodes at the last level fill the level “from left to right”
Example: complete binary tree
Example: not complete binary tree
Example: not complete binary tree
Level order

Ordering the nodes of a tree **level-by-level** (and left-to-right in each level).
Nodes of a complete binary tree

• How many nodes does a complete binary tree have at each level?

• At most
  - 1 at level 0.
  - 2 at level 1.
  - 4 at level 2.
  - ... 
  - \(2^k\) at level \(k\).
Properties of binary trees

• The following hold:
  - $h + 1 \leq n \leq 2^{h+1} - 1$
  - $1 \leq n_E \leq 2^h$
  - $h \leq n_I \leq 2^h - 1$
  - $\log(n + 1) - 1 \leq h \leq n - 1$

• Where
  - $n$: number of all nodes
  - $n_I$: number of internal nodes
  - $n_E$: number of external nodes (leaves)
  - $h$: height
Properties of complete binary trees

\( h \leq \log n \)

- Very important property, the tree cannot be too “tall”!
- Why?
  - Any level \( l < h \) contains exactly \( 2^l \) nodes
  - Level \( h \) contains at least one node
  - So \( 1 + 2 + \ldots + 2^{h-1} + 1 = 2^h \leq n \)
  - And take logarithms on both sides
How do we represent a binary tree?
Sequential representation

Store the entries in an **array** at **level order**.

- Common for **complete trees**
- A lot of **space** is wasted for non-complete trees
  - missing nodes will have empty slots in the array
# How to find nodes

<table>
<thead>
<tr>
<th>To Find:</th>
<th>Use</th>
<th>Provided</th>
</tr>
</thead>
<tbody>
<tr>
<td>The left child of $A[i]$</td>
<td>$A[2i]$</td>
<td>$2i \leq n$</td>
</tr>
<tr>
<td>The right child of $A[i]$</td>
<td>$A[2i + 1]$</td>
<td>$2i + 1 \leq n$</td>
</tr>
<tr>
<td>The parent of $A[i]$</td>
<td>$A[i/2]$</td>
<td>$i &gt; 1$</td>
</tr>
<tr>
<td>The root</td>
<td>$A[1]$</td>
<td>$A$ is nonempty</td>
</tr>
<tr>
<td>Whether $A[i]$ is a leaf</td>
<td></td>
<td>$2i &gt; n$</td>
</tr>
</tbody>
</table>
Heaps

A binary tree is called a heap (σωρός) if

- It is complete, and
- each node is greater or equal than its children

(Sometimes this is called a max-heap, we can similarly define a min-heap)
Example
Heaps and priority queues

- Heaps are a common data structure for implementing **Priority Queues**
- The following operations are needed
  - find max
  - insert
  - remove max
  - create with data
- We need to **preserve the heap property** in each operation!
Find max

- Trivial, the max is always at the root
  - remember: we always preserve the heap property
- Complexity?
Inserting a new element

• The new element can only be inserted at the end
  - because a heap must be a complete tree

• Now all nodes except the last satisfy the heap property
  - to restore it: apply the bubble_up algorithm on the last node
Inserting a new element

bubble_up(node)

• **Before**
  - node might be **larger** than its parent
  - all other nodes satisfy the heap property

• **After**
  - all nodes satisfy the heap property

• **Algorithm**
  - if node > parent
    - **swap them** and call bubble_up(parent)
Example insertion
Example insertion

Inserting 15 and running bubble_up
Example insertion

Inserting 12 and running bubble_up
Complexity of insertion

• We travel the tree from the last node to the root
  - on each node: 1 step (constant time)

• So we need at most $O(h)$ steps
  - $h$ is the height of the tree
  - but $h \leq \log n$ on a complete tree

• So $O(\log n)$
  - the “complete” property is crucial!
Removing the max element

- We want to remove the root
  - but the heap must be a **complete** tree

- So **swap** the root with the **last** element
  - then remove the last element

- Now all nodes **except the root** satisfy the heap property
  - to restore it: apply the **bubble_down** algorithm on the root
Removing the max element

bubble_down(node)

• Before
  - node might be smaller than any of its children
  - all other nodes satisfy the heap property

• After
  - all nodes satisfy the heap property

• Algorithm
  - max_child = the largest child of node
  - If node < max_child
    - swap them and call bubble_down(max_child)
Example removal
Example removal

Removing element 9

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 |
|---|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
|-INF| 8 | 7 | 6 | 4 | 3 | 2 | 5 | 1 | | | | | | | | | | | | | | | | | | | | | | | | | | |

Removing 9 and restoring the heap property
Complexity of removal

- We travel a single path from the root to a leaf
- So we need at most $O(h)$ steps
  - $h$ is the height of the tree
- Again $O(\log n)$
  - again, having a complete tree is crucial
Building a heap from initial data

- What if we want to create a heap that contains some initial values?
  - we call this operation **heapify**

- “Naive” implementation:
  - Create an empty heap and insert elements one by one

- What is the complexity of this implementation?
  - We do $n$ inserts
  - Each insert is $O(\log n)$ (because of bubble_up)
  - So $O(n \log n)$ total

- Worst-case example?
  - sorted elements: each value with have to fully bubble_up to the root
Efficient heapify

- Better algorithm:
  - Visit all **internal nodes** in **reverse level order**
    - last internal node: $\frac{n}{2}$ (parent of the last leaf $n$)
    - first internal node: 1 (root)
  - Call `bubble_down` on each visited node

- Why does this work?
  - when we visit node, its **subtree is already a heap**
    - except from node **itself** (the precondition of `bubble_down`)
  - So `bubble_down` restores the heap property **in the subtree**
  - After processing the root, the whole tree is a heap
Heapify example
Heapify example

Visit internal nodes in inverse level order, call bubble_down.
Complexity of heapify

• We call `bubble_down` \(\frac{n}{2}\) times
  - So \(O(n \log n)\)?

• But this is only an upper-bound
  - `bubble_down` is faster closer to the leaves
  - and most nodes live there!
  - we might be over-approximating the number of steps
Complexity of heapify

• More careful calculation of the number of steps:
  - If node is at level $l$, bubble_down takes at most $h - l$ steps.
  - At most $2^l$ nodes at this level, so $(h - l)2^l$ steps for level $l$.
  - For the whole tree: $\sum_{l=0}^{h-1} (h - l)2^l$.
  - This can be shown to be less than $2n$ (exercise if you're curious).

• So we get worst-case $O(n)$ complexity.
Efficient vs naive heapify

• For `naive_heapify` we found $O(n \log n)$
  - maybe we are also over-approximating?

• No: in the worst-case (sorted elements) we really need $n \log n$ steps
  - try to compute the exact number of steps

• The difference:
  - `bubble_up` is faster closer to the root, but few nodes live there
  - `bubble_down` is faster closer to the leaves, and most nodes live there

• Note: in the average-case, the naive version is also $O(n)$
Implementing ADTPriorityQueue

Types

```c
// Eνα PriorityQueue είναι pointer σε αυτό το struct

struct priority_queue {
    Vector vector;               // Τα δεδομένα, σε Vector για μεταβλη
    CompareFunc compare;         // Η διάταξη
    DestroyFunc destroy_value;   // Συνάρτηση που καταστρέφει ένα στοι
};
```
ADTPriorityQueue implementation

Types.

```c
// Eva PriorityQueue είναι pointer σε αυτό το struct

struct priority_queue {
    Vector vector; // Τα δεδομένα, σε Vector για μεταβλή
    CompareFunc compare; // Η διάταξη
    DestroyFunc destroy_value; // Συνάρτηση που καταστρέφει ένα στοι
};
```
ADTPriorityQueue implementation

Finding the max is trivial.

```c
Pointer pqueue_max(PriorityQueue pqueue) {
    return node_value(pqueue, 1);   // root
}
```
ADTPriorityQueue implementation

For `pqueue_insert`, the non-trivial part is `bubble_up`.

```java
// Αποκαθιστά την ιδιότητα του σωρού.
// Πριν: όλοι οι κόμβοι ικανοποιούν την ιδιότητα του σωρού, εκτός από
//      τον node που μπορεί να είναι _μεγαλύτερος_ από τον πατέρα του
// Μετά: όλοι οι κόμβοι ικανοποιούν την ιδιότητα του σωρού.

static void bubble_up(PriorityQueue pqueue, int node) {
    // Αν φτάσαμε στη ρίζα, σταματάμε
    if (node == 1)
        return;

    int parent = node / 2;        // 0 πατέρας του κόμβου. Τα node ids
    // Αν ο πατέρας έχει μικρότερη τιμή από τον κόμβο, swap και συνεχ
    if (pqueue->compare(node_value(pqueue, parent), node_value(pqueue
                      node_swap(pqueue, parent, node);
        bubble_up(pqueue, parent);
    }
```
ADTPriorityQueue implementation

// Πριν: όλοι οι κόμβοι ικανοποιούν την ιδιότητα του σωρού, εκτός από node που μπορεί να είναι _μικρότερος_ από κάποιο από τα παιδ // Μετά: όλοι οι κόμβοι ικανοποιούν την ιδιότητα του σωρού.

```java
static void bubble_down(PriorityQueue pqueue, int node) {
    // βρίσκουμε τα παιδιά του κόμβου (αν δεν υπάρχουν σταματάμε)
    int left_child = 2 * node;
    int right_child = left_child + 1;
    int size = pqueue_size(pqueue);
    if (left_child > size)
        return;
    // βρίσκουμε το μέγιστο από τα 2 παιδιά
    int max_child = left_child;
    if (right_child <= size && pqueue->compare(node_value(pqueue, left_child), node_value(pqueue, right_child))
        max_child = right_child;
    // Αν ο κόμβος είναι μικρότερος από το μέγιστο παιδί, swap και συ
    if (pqueue->compare(node_value(pqueue, node), node_value(pqueue, max_child))
        node_swap(pqueue, node, max_child);
    bubble_down(pqueue, max_child);
}
```
Other possible representations

<table>
<thead>
<tr>
<th>Operation</th>
<th>Heap</th>
<th>Sorted List</th>
<th>Unsorted Vector</th>
</tr>
</thead>
<tbody>
<tr>
<td>pqueue_create</td>
<td>$O(n)$</td>
<td>$O(n \log n)$</td>
<td>$O(1)$</td>
</tr>
<tr>
<td>pqueue_remove</td>
<td>$O(\log n)$</td>
<td>$O(1)$</td>
<td>$O(n)$</td>
</tr>
<tr>
<td>pqueue_insert</td>
<td>$O(\log n)$</td>
<td>$O(n)$</td>
<td>$O(1)$</td>
</tr>
</tbody>
</table>

All of them have some advantage

- Heaps provide a great compromise between insertions and removals
Using ADTPriorityQueue for sorting

- We can easily sort data using ADTPriorityQueue
  - create a priority queue with the data
  - remove elements in sorted order

- When ADTPriorityQueue is implemented by a heap
  - this algorithm is called heapsort
  - and runs in time $O(n \log n)$
Readings


• R. Sedgewick. *Αλγόριθμοι σε C*. Κεφ. 5 και 9.

Proofs of given statements can be found in the following book: