Binary Trees, Heaps

Κ08 Δομές Δεδομένων και Τεχνικές Προγραμματισμού
Κώστας Χατζηκοκολάκης
Binary trees

A binary tree (δυαδικό δέντρο) is a set of nodes such that:

• Exactly one node is called the root
• All nodes except the root have exactly one parent
• Each node has at most two children
  - and the are ordered: called left and right
Example: a binary tree
Example: a different binary tree

Whether a child is left or right matters.
Terminology

- **path**: sequence of nodes traversing from parent to child (or vice-versa)
- **length** of a path: number of nodes - 1 (= number of “moves” it contains)
- **siblings**: children of the same parent
- **descendants**: nodes reached by travelling downwards along any path
- **ancestors**: nodes reached by travelling upwards towards the root
- **leaf / external node**: a node without children
- **internal node**: a node with children
Terminology

• Nodes tree can be arranged in **levels / depths**:
  - The root is at **level 0**
  - Its children are at **level 1**, their children are at **level 2**, etc.

• Note: node level = length of the (unique) path from the root to that node

• **height** of the tree: the largest depth of any node

• **subtree** rooted at a node: the tree consisting of that node and its descendants
Complete binary trees

A binary tree is called **complete** (πλήρες) if

- All levels except the last are "**full**" (have the maximum number of nodes)
- The nodes at the last level fill the level “from left to right”
Example: complete binary tree
Example: not complete binary tree
Example: not complete binary tree

![Example: not complete binary tree](image-url)
Level order

Ordering the nodes of a tree **level-by-level** (and left-to-right in each level).
Nodes of a complete binary tree

• How many nodes does a complete binary tree have at each level?
  • At most
    - 1 at level 0.
    - 2 at level 1.
    - 4 at level 2.
    - ...
    - $2^k$ at level $k$. 
Properties of binary trees

- The following hold:
  - \( h + 1 \leq n \leq 2^{h+1} - 1 \)
  - \( 1 \leq n_E \leq 2^h \)
  - \( h \leq n_I \leq 2^h - 1 \)
  - \( \log(n + 1) - 1 \leq h \leq n - 1 \)

- Where
  - \( n \): number of all nodes
  - \( n_I \): number of internal nodes
  - \( n_E \): number of external nodes (leaves)
  - \( h \): height
Properties of complete binary trees

\[ h \leq \log n \]

- Very important property, the tree cannot be too “tall”!
- Why?
  - Any level \( l < h \) contains exactly \( 2^l \) nodes
  - Level \( h \) contains at least one node
  - So \( 1 + 2 + \ldots + 2^{h-1} + 1 = 2^h \leq n \)
  - And take logarithms on both sides
How do we represent a binary tree?
Sequential representation

Store the entries in an **array** at **level order**.

![Array representation of a tree](image)

- Common for **complete trees**
- A lot of **space** is wasted for non-complete trees
  - missing nodes will have empty slots in the array
# How to find nodes

<table>
<thead>
<tr>
<th>To Find:</th>
<th>Use</th>
<th>Provided</th>
</tr>
</thead>
<tbody>
<tr>
<td>The left child of $A[i]$</td>
<td>$A[2i]$</td>
<td>$2i \leq n$</td>
</tr>
<tr>
<td>The right child of $A[i]$</td>
<td>$A[2i + 1]$</td>
<td>$2i + 1 \leq n$</td>
</tr>
<tr>
<td>The parent of $A[i]$</td>
<td>$A[i/2]$</td>
<td>$i &gt; 1$</td>
</tr>
<tr>
<td>The root</td>
<td>$A[1]$</td>
<td>$A$ is nonempty</td>
</tr>
<tr>
<td>Whether $A[i]$ is a leaf</td>
<td></td>
<td>$2i &gt; n$</td>
</tr>
</tbody>
</table>
Heaps

A binary tree is called a heap (σωρός) if

• It is complete, and

• each node is greater or equal than its children

(Sometimes this is called a max-heap, we can similarly define a min-heap)
Example
Heaps and priority queues

• Heaps are a common data structure for implementing Priority Queues

• The following operations are needed
  - find max
  - insert
  - remove max
  - create with data

• We need to preserve the heap property in each operation!
Find max

- Trivial, the max is always at the root
  - remember: we always preserve the heap property

- Complexity?
Inserting a new element

• The new element can only be inserted at the end
  - because a heap must be a complete tree

• Now all nodes except the last satisfy the heap property
  - to restore it: apply the bubble_up algorithm on the last node
Inserting a new element

bubble_up(node)

• **Before**
  - node might be *larger* than its parent
  - all other nodes satisfy the heap property

• **After**
  - all nodes satisfy the heap property

• **Algorithm**
  - if node > parent
    - swap them and call bubble_up(parent)
Example insertion
Example insertion

Inserting 15 and running bubble_up
Example insertion

Inserting 12 and running bubble_up
Complexity of insertion

- We travel the tree from the last node to the root
  - on each node: 1 step (constant time)

- So we need at most $O(h)$ steps
  - $h$ is the height of the tree
  - but $h \leq \log n$ on a complete tree

- So $O(\log n)$
  - the “complete” property is crucial!
Removing the max element

- We want to remove the root
  - but the heap must be a **complete** tree

- So **swap** the root with the **last** element
  - then remove the last element

- Now all nodes **except the root** satisfy the heap property
  - to restore it: apply the **bubble_down** algorithm on the root
Removing the max element

bubble_down(node)

• Before
  - node might be smaller than any of its children
  - all other nodes satisfy the heap property

• After
  - all nodes satisfy the heap property

• Algorithm
  - max_child = the largest child of node
  - If node < max_child
    - swap them and call bubble_down(max_child)
Example removal

\[ \begin{array}{cccccccccccccccccccccccccccccccc}
\text{-INF} & 9 & 8 & 6 & 7 & 3 & 2 & 5 & 1 & 4 \\
\end{array} \]
Example removal

Removing element 9

Removing 9 and restoring the heap property
Complexity of removal

• We travel a single path from the root to a leaf
• So we need at most $O(h)$ steps
  - $h$ is the height of the tree
• Again $O(\log n)$
  - again, having a complete tree is crucial
Building a heap from initial data

• What if we want to create a heap that contains some initial values?
  - we call this operation heapify

• “Naive” implementation:
  - Create an empty heap and insert elements one by one

• What is the complexity of this implementation?
  - We do \( n \) inserts
  - Each insert is \( O(\log n) \) (because of bubble_up)
  - So \( O(n \log n) \) total

• Worst-case example?
  - sorted elements: each value with have to fully bubble_up to the root
Efficient heapify

• Better algorithm:
  - Visit all **internal nodes** in **reverse level order**
    ◦ last internal node: \( \frac{n}{2} \) (parent of the last leaf \( n \))
    ◦ first internal node: 1 (root)
  - Call `bubble_down` on each visited **node**

• Why does this work?
  - when we visit **node**, its **subtree is already a heap**
    ◦ except from **node itself** (the precondition of `bubble_down`)
  - So `bubble_down` restores the heap property **in the subtree**
  - After processing the root, the whole tree is a heap
Heapify example
Visit internal nodes in inverse level order, call bubble_down.
## Complexity of heapify

- We call `bubble_down` \( \frac{n}{2} \) times
  - So \( O(n \log n) \)?

- But this is only an **upper-bound**
  - `bubble_down` is faster **closer to the leaves**
  - and **most nodes** live there!
  - we might be over-approximating the number of steps
Complexity of heapify

- More careful calculation of the number of steps:
  - If node is at level $l$, `bubble_down` takes at most $h - l$ steps
  - At most $2^l$ nodes at this level, so $(h - l)2^l$ steps for level $l$
  - For the whole tree: $\sum_{l=0}^{h-1} (h - l)2^l$
    - This can be shown to be less than $2n$ (exercise if you're curious)
- So we get worst-case $O(n)$ complexity
Efficient vs naive heapify

- For naive_heapify we found $O(n \log n)$
  - maybe we are also over-approximating?
- No: in the worst-case (sorted elements) we really need $n \log n$ steps
  - try to compute the exact number of steps
- The difference:
  - bubble_up is faster closer to the root, but few nodes live there
  - bubble_down is faster closer to the leaves, and most nodes live there
- Note: in the average-case, the naive version is also $O(n)$
Implementing ADTPriorityQueue

Types

// Eνα PriorityQueue είναι pointer σε αυτό το struct

struct priority_queue {
    Vector vector;       // Τα δεδομένα, σε Vector για μεταβλη
    CompareFunc compare; // Η διάταξη
    DestroyFunc destroy_value; // Συνάρτηση που καταστρέφει ένα στοι
};
ADTPriorityQueue implementation

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};
ADTPriorityQueue implementation

Finding the max is trivial.

```c
Pointer pqueue_max(PriorityQueue pqueue) {
    return node_value(pqueue, 1); // root
}
```
ADTPriorityQueue implementation

For pqueue_insert, the non-trivial part is bubble_up.

```java
// Αποκαθιστά την ιδιότητα του σωρού.
// Πριν: όλοι οι κόμβοι ικανοποιούν την ιδιότητα του σωρού, εκτός από
// τον node που μπορεί να είναι _μεγαλύτερος_ από τον πατέρα το
// Μετά: όλοι οι κόμβοι ικανοποιούν την ιδιότητα του σωρού.

static void bubble_up(PriorityQueue pqueue, int node) {
    // Αν φτάσαμε στη ρίζα, σταματάμε
    if (node == 1)
        return;

    int parent = node / 2;       // O πατέρας του κόμβου. Ta node ids

    // Αν ο πατέρας έχει μικρότερη τιμή από τον κόμβο, swap και συνεχ
    if (pqueue->compare(node_value(pqueue, parent), node_value(pqueue
        node_swap(pqueue, parent, node);
    } bubble_up(pqueue, parent);
```
ADTPriorityQueue implementation

// Πριν: όλοι οι κόμβοι ικανοποιούν την ιδιότητα του σωρού, εκτός από
// node που μπορεί να είναι _μικρότερος_ από κάποιο από τα παιδ
// Μετά: όλοι οι κόμβοι ικανοποιούν την ιδιότητα του σωρού.

static void bubble_down(PriorityQueue pqueue, int node) {
    // βρίσκουμε τα παιδιά του κόμβου (αν δεν υπάρχουν σταματάμε)
    int left_child = 2 * node;
    int right_child = left_child + 1;
    int size = pqueue_size(pqueue);
    if (left_child > size)
        return;

    // βρίσκουμε το μέγιστο από τα 2 παιδιά
    int max_child = left_child;
    if (right_child <= size && pqueue->compare(node_value(pqueue, lef
        max_child = right_child;

    // Αν ο κόμβος είναι μικρότερος από το μέγιστο παιδί, swap και συ
    if (pqueue->compare(node_value(pqueue, node), node_value(pqueue,
        node_swap(pqueue, node, max_child);
        bubble_down(pqueue, max_child);
}
}
## Other possible representations

<table>
<thead>
<tr>
<th>Operation</th>
<th>Heap</th>
<th>Sorted List</th>
<th>Unsorted Vector</th>
</tr>
</thead>
<tbody>
<tr>
<td><code>pqueue_create</code> (with data)</td>
<td>$O(n)$</td>
<td>$O(n \log n)$</td>
<td>$O(1)$</td>
</tr>
<tr>
<td><code>pqueue_remove</code></td>
<td>$O(\log n)$</td>
<td>$O(1)$</td>
<td>$O(n)$</td>
</tr>
<tr>
<td><code>pqueue_insert</code></td>
<td>$O(\log n)$</td>
<td>$O(n)$</td>
<td>$O(1)$</td>
</tr>
</tbody>
</table>

All of them have **some** advantage

- Heaps provide a great compromise between insertions and removals
Using ADTPriorityQueue for sorting

• We can easily sort data using ADTPriorityQueue
  - create a priority queue with the data
  - remove elements in sorted order

• When ADTPriorityQueue is implemented by a heap
  - this algorithm is called heapsort
  - and runs in time $O(n \log n)$
Readings


- R. Sedgewick. Αλγόριθμοι σε C. Κεφ. 5 και 9.

Proofs of given statements can be found in the following book: