Graphs (Γράφοι)

Κ08 Δομές Δεδομένων και Τεχνικές Προγραμματισμού

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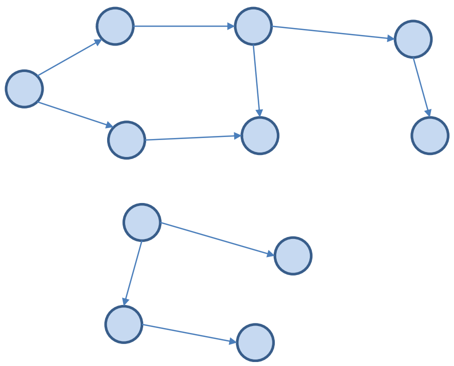
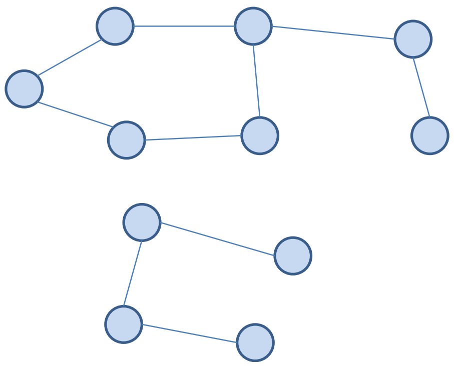
Graphs

- **Graphs** are collections of nodes in which various pairs are connected by line segments. The nodes are usually called **vertices** (**κορυφές**) and the line segments **edges (ακμές).**
- Graphs are **more general than trees**. Graphs are allowed to have cycles and can have more than one connected component.
- Some authors use the terms **nodes (κόμβοι)** and **arcs (τόξα)** instead of vertices and edges.

Example of Graphs (Directed)



Example of Graphs (Undirected)



Examples of Graphs

- Transportation networks
- **Interesting problem**: What is the path with one or more stops of shortest overall distance connecting a starting city and a destination city?

Examples

- A network of oil pipelines
- **Interesting problem**: What is the maximum possible overall flow of oil from the source to the destination?

Examples

- The Internet
- **Interesting problem**: Deliver an e-mail from user A to user B

Examples

- The Web
- **Interesting problem**: What is the PageRank of a Web site?

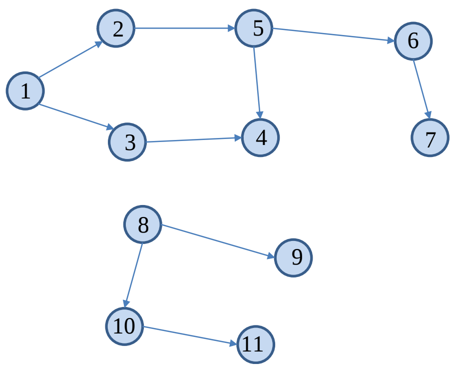
Examples

- The Facebook social network
- **Interesting problem**: Are John and Mary connected? What interesting clusters exist?

Formal Definitions

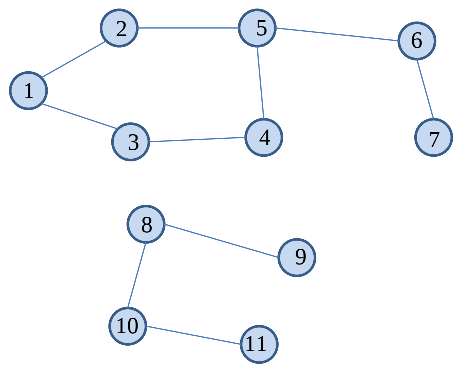
- A **graph** $G = (V, E)$ consists of a set of **vertices** V and a set of **edges** E , where the edges in E are formed from pairs of **distinct** vertices in V.
- If the edges have directions then we have a **directed graph (κατευθυνόμενο γράφο)** or **digraph**. In this case edges are ordered pairs of vertices e.g., (u, v) and are called **directed**. If (u, v) is a directed edge then u is called its **origin** and v is called its **destination**.
- If the edges do not have directions then we have an **undirected graph (μη-κατευθυνόμενος γράφο)**. In this case edges are unordered pairs of vertices e.g., $\{u, v\}$ and are called **undirected**.
- For simplicity, we will use the directed pair notation noting that in the undirected case (u, v) is the same as (v, u) .
- When we say simply graph, we will mean an undirected graph.

Example of a Directed Graph



$G = (V, E)$
$V = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11$
$E = (1, 2), (1, 3), (2, 5), (3, 4), (5, 4), (5, 6), (6, 70, (8, 9), (8, 10), (10, 11$

Example of an Undirected Graph



$G = (V, E)$
$V = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11$
$E = (1, 2), (1, 3), (2, 5), (3, 4), (5, 4), (5, 6), (6, 70, (8, 9), (8, 10), (10, 11$

More Definitions

- Two different vertices v_i, v_j in a graph $G = (V, E)$ are said to be **adjacent (γειτονικές)** if there exists an edge $(v_i, v_j) \in E$.
- An edge is said to be **incident (προσπίπτουσα)** on a vertex if the vertex is one of the edge's endpoints.
- A **path (μονοπάτι)** p in a graph $G = (V, E)$, is a sequence of vertices of V of the form $p = v_1 v_2 \ldots v_n, (n \geq 2)$ in which each vertex v_i, is adjacent to the next one v_{i+1} (for $1 \leq i \leq n-1$).
- The **length** of a path is the number of edges in it.
- A path is **simple** if each vertex in the path is distinct.
- A **cycle** is a path $p = v_1 v_2 \ldots v_n$ of length greater than one that begins and ends at the same vertex (i.e., $v_1 = v_n$).

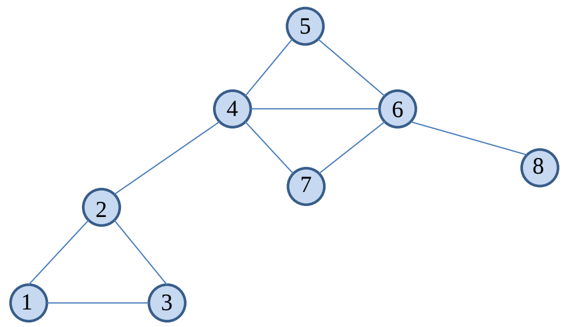
Definitions

- A **directed path** is a path such that all edges are directed and are traversed along their direction.
- A **directed cycle** is similarly defined.

Definitions

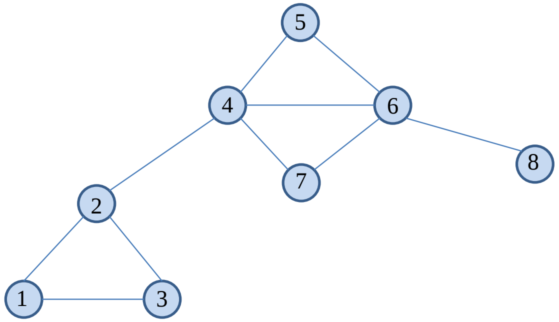
- A **simple cycle** is a path that travels through three or more **distinct** vertices and connects them into a loop.

Example



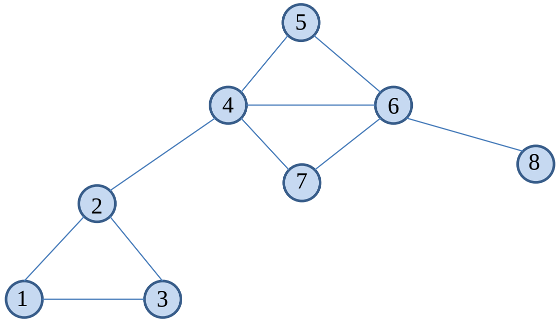
Four simple cycles: (1,2,3,1) (4,5,6,7,4) (4,5,6,4) (4,6,7,4)

Example



Two non-simple cycles: (1,2,1) (4,5,6,4,7,6,4)

Example



A path that is not a cycle: (1,2,4,6,8)

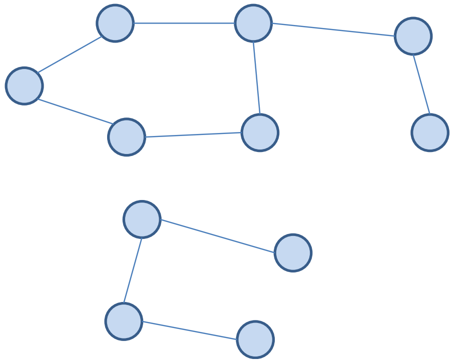
Connectivity and Components

- Two vertices in a graph $G = (V, E)$ are said to be **connected (συνδεδεμένες)** if there is a path from the first to the second in G
- Formally, if $x \in V$ and $y \in V$, where $x \neq y$, then x and y are **connected** if there exists a path $p = v_1 v_2 \ldots v_n \in G$ in such that $x = v_1$ and $y = v_n$

Connectivity and Components

- In the graph $G = (V, E)$, a **connected component (συνεκτική συνιστώσα)** is a subset S of the vertices V that are all connected to one another.
- A connected component S of G is a **maximal connected component (μέγιστη συνεκτική συνιστώσα)** provided there is no bigger subset T of vertices in V such that T properly contains S and such that T itself is a connected component of G.
- An undirected graph G can always be separated into maximal connected components $S_1, S_2, \ldots;, S_n$ such that $S_i \cap S_j = \varnothing$ whenever $i \neq j$.

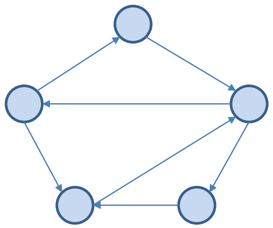
Example of Undirected Graph and its Separation into Two Maximal Connected Components



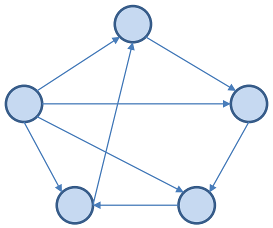
Connectivity and Components in Directed Graphs

- A subset S of vertices in a **directed** graph G is **strongly connected (ισχυρά συνεκτικό)** if for each pair of distinct vertices (v_i, v_j) in S, v_i is connected to v_j **and** v_j is connected to v_i.
- A subset S of vertices in a **directed** graph G is **weakly connected (ασθενώς συνεκτικό)** if for each pair of distinct vertices (v_i, v_j)in S, v_i is connected to v_j **or** v_j is connected to v_i.

Example: A Strongly Connected Digraph



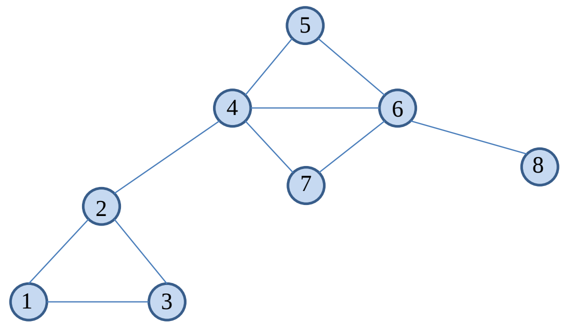
Example: A Weakly Connected Digraph



Degree in Undirected Graphs

- In an undirected graph G the **degree** (**βαθμός**) of vertex x is the number of edges e in which x is one of the endpoints of e.
- The degree of a vertex x is denoted by $\deg(x)$.

Example



The degree of node 1 is 2.
The degree of node 4 is 4.
The degree of node 8 is 1.

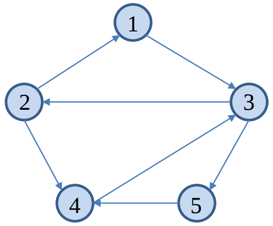
Predecessors and Successors in Directed Graphs

- If x is a vertex in a **directed** graph $G = (V, E)$ then the set of **predecessors (προηγούμενων)** of x denoted by $\mathrm{Pred}(x)$ is the set of all vertices $y \in V$ such that $(y, x) \in E$.
- Similarly the set of **successors (επόμενων)** of x denoted by $\mathrm{Succ}(x)$ is the set of all vertices $y \in V$ such that $(x, y) \in E$.

In-Degree and Out-Degree in Directed Graphs

- The **in-degree** of a vertex x is the number of predecessors of x
- The **out-degree** of a vertex x is the number of successors of x
- We can also define the in-degree and the out-degree by referring to the **incoming** and **outgoing** edges of a vertex.
- The in-degree and out-degree of a vertex x are denoted by $\mathrm{indeg}(x)$ and $\mathrm{outdeg}(x)$ respectively.

Example



The in-degree of node 4 is 2. The out-degree of node 4 is 1.

Proposition

- If G is an undirected graph with m edges, then

$$\sum_{v \in G} \deg(v) =$$

.

- Proof?
  - Each edge is counted twice

Proposition

- If is a directed graph with edges, then

$$\sum_{v \in G} \mathrm{indeg}(v) = \sum_{v \in G} \mathrm{outdeg}(v) = m$$

- Proof?
  - Each edge is counted once

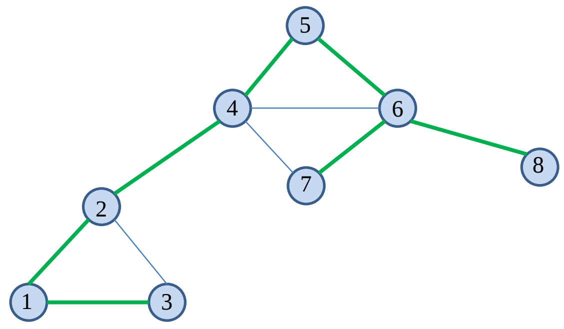
Proposition

- Let G be a graph with n vertices and m edges. If G is undirected, then $m \leq \frac{n(n-1)}{2}$ and if G is directed, then $m \leq n(n-1)$.
- Proof?
  - If G is undirected then the maximum degree of a vertex is $n-1$. Therefore, from the previous proposition about the sum of the degrees, we have $2m \leq n(n-1)$.
  - If G is directed then the maximum in-degree of a vertex is $n-1$. Therefore, from the previous proposition about the sum of the in-degrees, we have $m \leq n(n-1)$.

More definitions

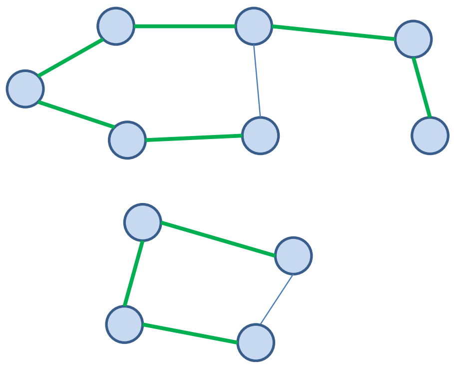
- A **subgraph (υπογράφος)** of a graph G is a graph H whose vertices and edges are subsets of the vertices and edges of G respectively.
- A **spanning subgraph (υπογράφος επικάλυψης)** of G is a subgraph of G that contains all the vertices of G.
- A **forest (δάσος)** is a graph without cycles.
- A **free tree (ελεύθερο δένδρο)** is a connected forest i.e., a connected graph without cycles. The trees that we studied in earlier lectures are **rooted trees (δένδρα με ρίζα)** and they are different than free trees.
- A **spanning tree (δένδρο επικάλυψης)** of a graph is a spanning subgraph that is a free tree.

Example



The thick green lines define a spanning tree of the graph.

Example

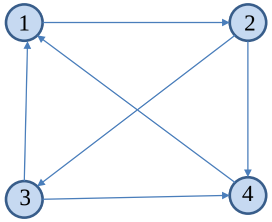


The thick green lines define a forest which consists of two free trees.

Graph Representations: Adjacency Matrices

- Let $G = (V, E)$ be a graph. Suppose we number the vertices in V as $v_1, v_2 \ldots v_n$
- The **adjacency matrix (πίνακας γειτνίασης)** corresponding to G is an $n \times n$ matrix such that $T[i, j] = 1$ if there is an edge $(v_i, v_j) \in E$, and $T[i, j] = 0$ if there is no such edge in E.

Example



A graph G

	1	2	3	4
1	0	1	0	0
2	0	0	1	1
3	1	0	0	1
4	1	0	0	0

The adjacency matrix for graph G

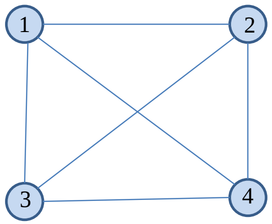
Adjacency Matrices

- The adjacency matrix of an **undirected graph** G is a **symmetric matrix** i.e., $T[i, j] = T[j, i]$ for all and in the range $1 \leq i, j \leq n$
- The adjacency matrix for a **directed graph** need not be symmetric.

Adjacency Matrices

- The **diagonal entries** in an adjacency matrix (of a directed or undirected graph) **are zero**, since graphs as we have defined them are not permitted to have looping self-referential edges that connect a vertex to itself.

Example



An undirected graph G

	1	2	3	4
1	0	1	1	1
2	1	0	1	1
3	1	1	0	1
4	1	1	1	0

The adjacency matrix for graph G

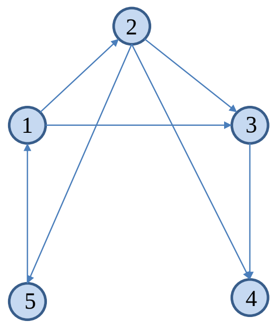
Adjacency Sets

- Another way to define a graph $G = (V, E)$ is to specify **adjacency sets (σύνολα γειτνίασης)** for each vertex in V.
- Let V_x stand for the set of all vertices **adjacent** to x in an undirected graph G or the set of all vertices that are **successors** of x in a directed graph G.
- If we give both the vertex set v and the collection $A = \{V_x | x \in V\}$ of adjacency sets for each vertex in then we have given enough information to define the graph G.

Graph Representations: Adjacency Lists

- Another family of representations for a graph uses **adjacency lists (λίστες γειτνίασης)** to represent the adjacency set V_x for each vertex x in the graph.

Example Directed Graph

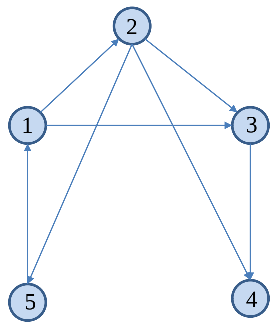


A directed graph G

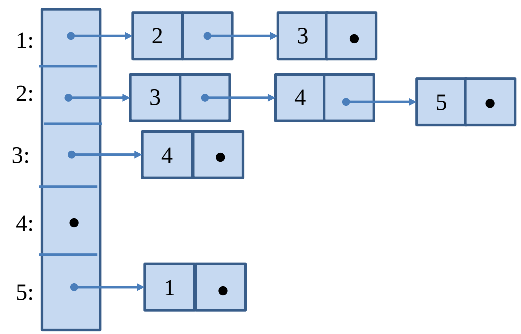
Vertex Number	Out Degree	Adjacency list
1	2	2 3
2	3	3 4 5
3	1	4
4	0	
5	1	1

The **sequential** adjacency lists for graph G. Notice that vertices are listed in their **natural order**.

Example Directed Graph

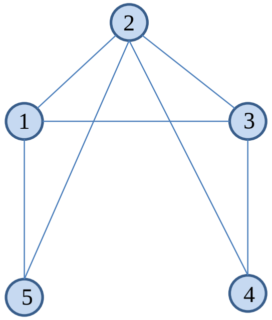


A directed graph G



The **linked** adjacency lists for graph G. Notice that vertices in a list are organized according to their **natural order.**

Example Undirected Graph



An undirected graph G

Vertex Number	Degree	Adjacency list
1	3	2 3 5
2	4	1 3 4 5
3	3	1 2 4
4	2	2 4
5	2	1 2

The sequential adjacency lists for graph G

Graph Searching

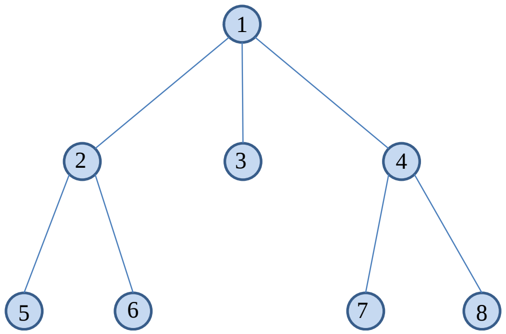
- To search a graph G, we need to visit all vertices of G in some systematic order.
- Each vertex v can be a structure with a `bool` valued member v. Visited which is initially `false` for all vertices of G. When we visit v, we will set it to `true`.

An Algorithm for Graph Searching

```
// Ψευδοκώδικας, επίσκεψη όλων των κόμβων του γράφου

void graph_search(G) {
    Let G = (V,E) be a graph
    Let C be an empty container

    for (each vertex x in V) {
        x.visited = false;
    }
    Insert v into C;

    while (C is non-empty) {
        Remove a vertex x from container C;
        if (!x.visited) {
            visit(x);
            x.visited = true;
            for (each vertex w adjacent to x) {
                if (!w.visited))
                   Insert w into C;
            }
        }
    }
}
```


Graph Searching

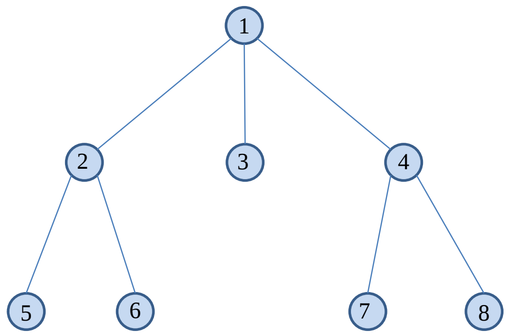
Interesting case: the container C is a stack.



In what order vertices are visited?

Graph Searching

Eg. the container C is a stack.



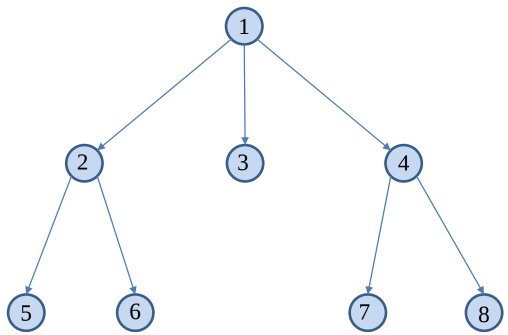
The vertices are visited in the order 1, 4, 8, 7, 3, 2, 6, 5.

Depth-First Search (DFS)

- When C is a **stack**, the tree in the previous example is searched in **depth-first order**.
- **Depth-first search (αναζήτηση πρώτα κατά βάθος)** at a vertex always goes down (by visiting unvisited children) before going across (by visiting unvisited brothers and sisters).
- Depth-first search of a graph is analogous to a **pre-order traversal** of an ordered tree.

Graph Searching

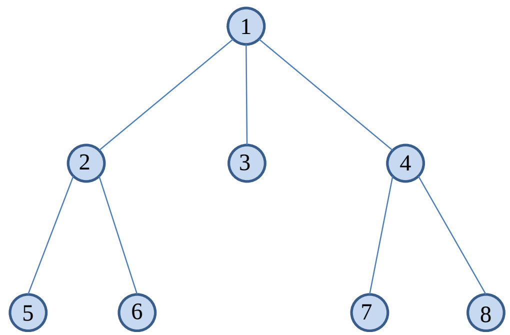
Another interestg case: the container C is a queue.



What is the order vertices are visited?

Graph Searching

Another interestg case: the container C is a queue.

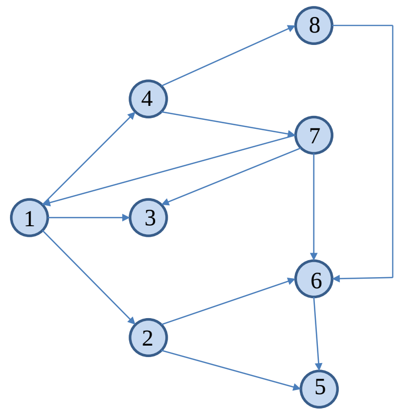


The vertices are visited in the order 1, 2, 3, 4, 5, 6, 7 and 8.

Breadth-First Search (BFS)

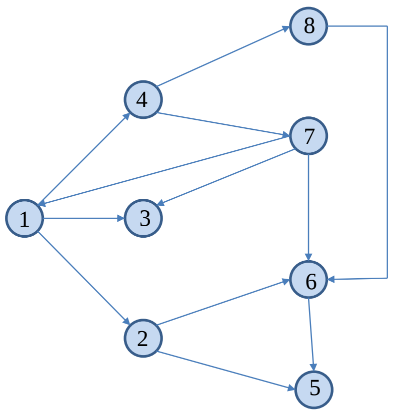
- When C is a **queue**, the tree in the previous example is searched in **breadth-first order**.
- **Breadth-first search (αναζήτηση πρώτα κατά πλάτος)** at a vertex always goes broad before going deep.
- Breadth-first traversal of a graph is analogous to a traversal of an ordered tree that visits the nodes of the tree in **level-order**.
- BFS subdivides the vertices of a graph in **levels**. The starting vertex is at level 0, then we have the vertices adjacent to the starting vertex at level 1, then the vertices adjacent to these vertices at level 2 etc.

Example



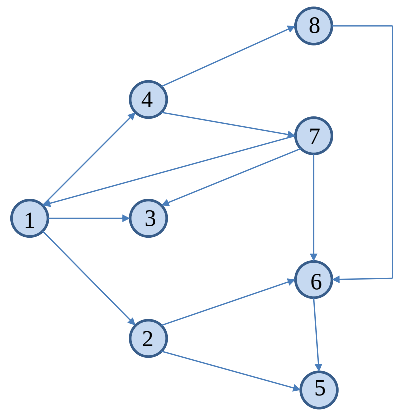
What is the order of visiting vertices for DFS?

Example



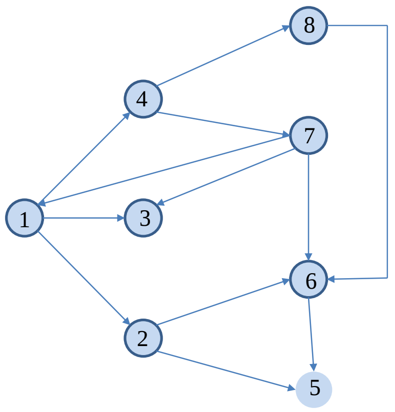
Depth-first search visits the vertices in the order 1, 4, 8, 6, 5, 7, 3 and 2

Example



What is the order of visit for BFS?

Example



Breadth-first search visits the vertices in the order 1, 2, 3, 4, 5, 6, 7 and 8.

Exhaustive Search

- Either the stack version or the queue version of the algorithm `GraphSearch` will visit every vertex in a graph G provided that G consists of a single strongly connected component.
- If this is not the case, then we can enumerate all the vertices of G and run `GraphSearch` starting from each one of them in order to visit all the vertices of G.

Exhaustive Search

```
void graph_exhaustive_search(G) {
    Let G = (V,E) be a graph.
    for (each vertex v in G) {
        graph_search(G, v)
    }
}
```


Recursive DFS

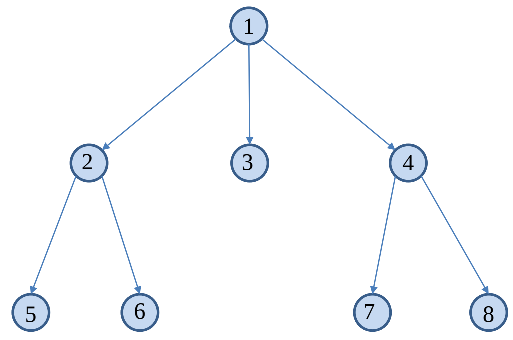
- DFS can be also written recursively
- The stack is essentially replaced by the **function call stack**

Recursive DFS

```
// Ψευδοκώδικας, επίσκεψη όλων των κόμβων του γράφου

void graph_dfs(G) {
    for (each vertex x in V) {
        x.visited = false;
    }
    for (each vertex x in V) {
        if (!x.visited))
            traverse(G, x);
    }
}

void traverse(G, x) {
    visit(x);
    x.visited = true;

    for (each vertex w adjacent to v) {
        if (!w.visited))
            traverse(G, w);
    }
}
```

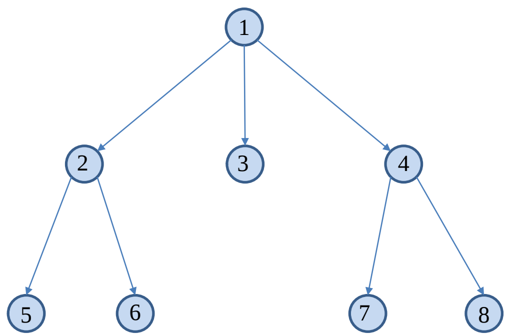

Example of Recursive DFS

What is the order vertices are visited?



Example

The vertices are visited in the order 1, 2, 5, 6, 3, 4, 7 and 8. This is different than the order we got when using a stack!



Complexity of DFS

- DFS as implemented above (with adjacency lists) on a graph with e edges and n vertices has complexity $O(n+e)$.
- To see why observe that on no vertex is `traverse` called more than once, because as soon as we call `traverse` with parameter x , we mark x visited and we never call `traverse` on a vertex that has previously been marked as visited.
- Thus, the total time spent going down the adjacency lists is proportional to the lengths of those lists, that is $O(e)$
- The initialization steps in `graph_dfs` have complexity $O(n)$
- Thus, the total complexity is $O(n+e)$

Complexity of DFS

- If DFS is implemented using an adjacency matrix, then its complexity will be $O(n^2)$.
- If the graph is **dense (πυκνός)**, that is, it has close to $O(n^2)$ edges the difference of the two implementations is minor as they would both run in $O(n^2)$ time.
- If the graph is **sparse (αραιός)**, that is, it has close to $O(n)$ edges, then the adjacency matrix approach would be much slower than the adjacency list approach.

Complexity of BFS

- BFS with adjacency lists has the same complexity as DFS i.e., $O(n + e)$.

Readings

- T. A. Standish. *Data Structures , Algorithms and Software Principles in C.* Chapter 10
- R. Kruse and C.L. Tondo and B. Leung. *Data Structures and Program Design in C.* 2nd edition. Chapter 11
- A. V. Aho, J. E. Hopcroft and J. D. Ullman. *Data Structures and Algorithms.* Chapters 6 and 7
- M. T. Goodrich, R. Tamassia and D. Mount. *Data Structures and Algorithms in C++.* 2nd edition. Chapter 13