Graphs (Γράφοι)

Κ08 Δομές Δεδομένων και Τεχνικές Προγραμματισμού
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Graphs

- **Graphs** are collections of nodes in which various pairs are connected by line segments. The nodes are usually called **vertices** (κορυφές) and the line segments **edges** (ακμές).

- Graphs are **more general than trees**. Graphs are allowed to have cycles and can have more than one connected component.

- Some authors use the terms **nodes** (κόμβοι) and **arcs** (τόξα) instead of vertices and edges.
Example of Graphs (Directed)
Example of Graphs (Undirected)
Examples of Graphs

• Transportation networks

• **Interesting problem**: What is the path with one or more stops of shortest overall distance connecting a starting city and a destination city?
Examples

• A network of oil pipelines

• Interesting problem: What is the maximum possible overall flow of oil from the source to the destination?
Examples

- The Internet

- **Interesting problem**: Deliver an e-mail from user A to user B
Examples

- The Web
- **Interesting problem**: What is the PageRank of a Web site?
Examples

• The Facebook social network

• *Interesting problem*: Are John and Mary connected? What interesting clusters exist?
Formal Definitions

- A graph $G = (V, E)$ consists of a set of vertices $V$ and a set of edges $E$, where the edges in $E$ are formed from pairs of distinct vertices in $V$.

- If the edges have directions then we have a directed graph (κατευθυνόμενο γράφο) or digraph. In this case edges are ordered pairs of vertices e.g., $(u, v)$ and are called directed. If $(u, v)$ is a directed edge then $u$ is called its origin and $v$ is called its destination.

- If the edges do not have directions then we have an undirected graph (μη-κατευθυνόμενος γράφο). In this case edges are unordered pairs of vertices e.g., $\{u, v\}$ and are called undirected.

- For simplicity, we will use the directed pair notation noting that in the undirected case $(u, v)$ is the same as $(v, u)$.

- When we say simply graph, we will mean an undirected graph.
Example of a Directed Graph

\[ G = (V, E) \]

\[ V = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\} \]

\[ E = \{(1, 2), (1, 3), (2, 5), (3, 4), (5, 4), (5, 6), (6, 70), (8, 9), (8, 10), (10, 11)\} \]
Example of an Undirected Graph

\[ G = (V, E) \]
\[ V = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 \]
\[ E = (1, 2), (1, 3), (2, 5), (3, 4), (5, 4), (5, 6), (6, 7), (8, 9), (8, 10), (10, 11) \]
More Definitions

• Two different vertices $v_i, v_j$ in a graph $G = (V, E)$ are said to be adjacent (γειτονικές) if there exists an edge $(v_i, v_j) \in E$.

• An edge is said to be incident (προσπίπτουσα) on a vertex if the vertex is one of the edge's endpoints.

• A path (μονοπάτι) $p$ in a graph $G = (V, E)$, is a sequence of vertices of $V$ of the form $p = v_1v_2 \ldots v_n, (n \geq 2)$ in which each vertex $v_i$, is adjacent to the next one $v_{i+1}$ (for $1 \leq i \leq n - 1$).

• The length of a path is the number of edges in it.

• A path is simple if each vertex in the path is distinct.

• A cycle is a path $p = v_1v_2 \ldots v_n$ of length greater than one that begins and ends at the same vertex (i.e., $v_1 = v_n$).
Definitions

• A **directed path** is a path such that all edges are directed and are traversed along their direction.

• A **directed cycle** is similarly defined.
Definitions

• A **simple cycle** is a path that travels through three or more **distinct** vertices and connects them into a loop.
Example

Four simple cycles: $(1, 2, 3, 1) \ (4, 5, 6, 7, 4) \ (4, 5, 6, 4) \ (4, 6, 7, 4)$
Example

Two non-simple cycles: (1,2,1) (4,5,6,4,7,6,4)
Example

A path that is not a cycle: (1, 2, 4, 6, 8)
Connectivity and Components

- Two vertices in a graph $G = (V, E)$ are said to be connected (συνδεδεμένες) if there is a path from the first to the second in $G$

- Formally, if $x \in V$ and $y \in V$, where $x \neq y$, then $x$ and $y$ are connected if there exists a path $p = v_1 v_2 \ldots v_n \in G$ in such that $x = v_1$ and $y = v_n$
Connectivity and Components

• In the graph $G = (V, E)$, a **connected component** (συνεκτική συνιστώσα) is a subset $S$ of the vertices $V$ that are all connected to one another.

• A connected component $S$ of $G$ is a **maximal connected component** (μέγιστη συνεκτική συνιστώσα) provided there is no bigger subset $T$ of vertices in $V$ such that $T$ properly contains $S$ and such that $T$ itself is a connected component of $G$.

• An undirected graph $G$ can always be separated into maximal connected components $S_1, S_2, \ldots, S_n$ such that $S_i \cap S_j = \emptyset$ whenever $i \neq j$. 
Example of Undirected Graph and its Separation into Two Maximal Connected Components
Connectivity and Components in Directed Graphs

• A subset $S$ of vertices in a directed graph $G$ is strongly connected (ισχυρά συνεκτικό) if for each pair of distinct vertices $(v_i, v_j)$ in $S$, $v_i$ is connected to $v_j$ and $v_j$ is connected to $v_i$.

• A subset $S$ of vertices in a directed graph $G$ is weakly connected (ασθενώς συνεκτικό) if for each pair of distinct vertices $(v_i, v_j)$ in $S$, $v_i$ is connected to $v_j$ or $v_j$ is connected to $v_i$. 
Example: A Strongly Connected Digraph
Example: A Weakly Connected Digraph
Degree in Undirected Graphs

• In an undirected graph $G$ the degree (βαθμός) of vertex $x$ is the number of edges $e$ in which $x$ is one of the endpoints of $e$.

• The degree of a vertex $x$ is denoted by $\text{deg}(x)$.
The degree of node 1 is 2.
The degree of node 4 is 4.
The degree of node 8 is 1.
Predecessors and Successors in Directed Graphs

• If $x$ is a vertex in a directed graph $G = (V, E)$ then the set of predecessors (προηγούμενων) of $x$ denoted by $\text{Pred}(x)$ is the set of all vertices $y \in V$ such that $(y, x) \in E$.

• Similarly the set of successors (επόμενων) of $x$ denoted by $\text{Succ}(x)$ is the set of all vertices $y \in V$ such that $(x, y) \in E$. 
In-Degree and Out-Degree in Directed Graphs

- The **in-degree** of a vertex \( x \) is the number of predecessors of \( x \).
- The **out-degree** of a vertex \( x \) is the number of successors of \( x \).
- We can also define the in-degree and the out-degree by referring to the **incoming** and **outgoing** edges of a vertex.
- The in-degree and out-degree of a vertex \( x \) are denoted by \( \text{indeg}(x) \) and \( \text{outdeg}(x) \) respectively.
The in-degree of node 4 is 2. The out-degree of node 4 is 1.
Proposition

- If $G$ is an undirected graph with $m$ edges, then
  \[ \sum_{v \in G} \deg(v) = m. \]

- Proof?
  - Each edge is counted twice
Proposition

• If $G$ is a directed graph with edges, then

$$\sum_{v \in G} \text{indeg}(v) = \sum_{v \in G} \text{outdeg}(v) = m$$

• Proof?
  - Each edge is counted once
Proposition

- Let $G$ be a graph with $n$ vertices and $m$ edges. If $G$ is undirected, then $m \leq \frac{n(n-1)}{2}$ and if $G$ is directed, then $m \leq n(n - 1)$.

- Proof?
  - If $G$ is undirected then the maximum degree of a vertex is $n - 1$. Therefore, from the previous proposition about the sum of the degrees, we have $2m \leq n(n - 1)$.
  - If $G$ is directed then the maximum in-degree of a vertex is $n - 1$. Therefore, from the previous proposition about the sum of the in-degrees, we have $m \leq n(n - 1)$. 
More definitions

- A subgraph (υπογράφος) of a graph $G$ is a graph $H$ whose vertices and edges are subsets of the vertices and edges of $G$ respectively.

- A spanning subgraph (υπογράφος επικάλυψης) of $G$ is a subgraph of $G$ that contains all the vertices of $G$.

- A forest (δάσος) is a graph without cycles.

- A free tree (ελεύθερο δένδρο) is a connected forest i.e., a connected graph without cycles. The trees that we studied in earlier lectures are rooted trees (δένδρα με ρίζα) and they are different than free trees.

- A spanning tree (δένδρο επικάλυψης) of a graph is a spanning subgraph that is a free tree.
Example

The thick green lines define a spanning tree of the graph.
Example

The thick green lines define a forest which consists of two free trees.
Graph Representations: Adjacency Matrices

- Let $G = (V, E)$ be a graph. Suppose we number the vertices in $V$ as $v_1, v_2 \ldots v_n$

- The **adjacency matrix** (πίνακας γειτνίασης) corresponding to $G$ is an $n \times n$ matrix such that $T[i, j] = 1$ if there is an edge $(v_i, v_j) \in E$, and $T[i, j] = 0$ if there is no such edge in $E$. 
Example

A graph $G$

The adjacency matrix for graph $G$

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
Adjacency Matrices

- The adjacency matrix of an **undirected graph** $G$ is a **symmetric matrix** i.e., $T[i, j] = T[j, i]$ for all and in the range $1 \leq i, j \leq n$

- The adjacency matrix for a **directed graph** need not be symmetric.
Adjacency Matrices

- The **diagonal entries** in an adjacency matrix (of a directed or undirected graph) **are zero**, since graphs as we have defined them are not permitted to have looping self-referential edges that connect a vertex to itself.
Example

An undirected graph $G$

The adjacency matrix for graph $G$

$$
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
1 & 0 & 1 & 1 & 1 \\
2 & 1 & 0 & 1 & 1 \\
3 & 1 & 1 & 0 & 1 \\
4 & 1 & 1 & 1 & 0 \\
\end{array}
$$
Adjacency Sets

• Another way to define a graph $G = (V, E)$ is to specify adjacency sets (σύνολα γειτνίασης) for each vertex in $V$.

• Let $V_x$ stand for the set of all vertices adjacent to $x$ in an undirected graph $G$ or the set of all vertices that are successors of $x$ in a directed graph $G$.

• If we give both the vertex set $v$ and the collection $A = \{V_x | x \in V\}$ of adjacency sets for each vertex in then we have given enough information to define the graph $G$. 
Another family of representations for a graph uses adjacency lists (λίστες γειτνίασης) to represent the adjacency set $V_x$ for each vertex $x$ in the graph.
Example Directed Graph

A directed graph $G$

<table>
<thead>
<tr>
<th>Vertex Number</th>
<th>Out Degree</th>
<th>Adjacency list</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>2 3</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>3 4 5</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

The **sequential** adjacency lists for graph $G$. Notice that vertices are listed in their **natural order**.
Example Directed Graph

A directed graph $G$

The **linked** adjacency lists for graph $G$. Notice that vertices in a list are organized according to their **natural order**.
Example Undirected Graph

An undirected graph $G$

<table>
<thead>
<tr>
<th>Vertex Number</th>
<th>Degree</th>
<th>Adjacency list</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
<td>2 3 5</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>1 3 4 5</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>1 2 4</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>2 4</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>1 2</td>
</tr>
</tbody>
</table>
Graph Searching

• To search a graph $G$, we need to visit all vertices of $G$ in some systematic order.

• Each vertex $v$ can be a structure with a `bool` valued member $v$. Visited which is initially `false` for all vertices of $G$. When we visit $v$, we will set it to `true`. 
An Algorithm for Graph Searching

void graph_search(G) {
    Let G = (V,E) be a graph
    Let C be an empty container

    for (each vertex x in V) {
        x.visited = false;
    }
    Insert v into C;

    while (C is non-empty) {
        Remove a vertex x from container C;
        if (!x.visited) {
            visit(x);
            x.visited = true;
            for (each vertex w adjacent to x) {
                if (!w.visited)
                    Insert w into C;
            }
        }
    }
}
Graph Searching

Interesting case: the container $C$ is a stack.

In what order vertices are visited?
Graph Searching

Eg. the container $C$ is a stack.

The vertices are visited in the order 1, 4, 8, 7, 3, 2, 6, 5.
Depth-First Search (DFS)

- When $C$ is a stack, the tree in the previous example is searched in depth-first order.

- Depth-first search (αναζήτηση πρώτα κατά βάθος) at a vertex always goes down (by visiting unvisited children) before going across (by visiting unvisited brothers and sisters).

- Depth-first search of a graph is analogous to a pre-order traversal of an ordered tree.
Another interesting case: the container $C$ is a queue.

What is the order vertices are visited?
Graph Searching

Another interesting case: the container $C$ is a queue.

The vertices are visited in the order 1, 2, 3, 4, 5, 6, 7 and 8.
Breadth-First Search (BFS)

- When $C$ is a queue, the tree in the previous example is searched in breadth-first order.

- **Breadth-first search (αναζήτηση πρώτα κατά πλάτος)** at a vertex always goes broad before going deep.

- Breadth-first traversal of a graph is analogous to a traversal of an ordered tree that visits the nodes of the tree in **level-order**.

- BFS subdivides the vertices of a graph in **levels**. The starting vertex is at level 0, then we have the vertices adjacent to the starting vertex at level 1, then the vertices adjacent to these vertices at level 2 etc.
Example

What is the order of visiting vertices for DFS?
Depth-first search visits the vertices in the order 1, 4, 8, 6, 5, 7, 3 and 2
What is the order of visit for BFS?
Example

Breadth-first search visits the vertices in the order 1, 2, 3, 4, 5, 6, 7 and 8.
Exhaustive Search

• Either the stack version or the queue version of the algorithm \texttt{GraphSearch} will visit every vertex in a graph \( G \) provided that \( G \) consists of a single strongly connected component.

• If this is not the case, then we can enumerate all the vertices of \( G \) and run \texttt{GraphSearch} starting from each one of them in order to visit all the vertices of \( G \).
Exhaustive Search

```c
void graph_exhaustive_search(G) {
    Let G = (V,E) be a graph.
    for (each vertex v in G) {
        graph_search(G, v)
    }
}
```
Recursive DFS

• DFS can be also written recursively
• The stack is essentially replaced by the **function call stack**
Recursive DFS

// Ψευδοκώδικας, επίσκεψη όλων των κόμβων του γράφου

```java
void graph_dfs(G) {
    for (each vertex x in V) {
        x.visited = false;
    }
    for (each vertex x in V) {
        if (!x.visited))
            traverse(G, x);
    }
}

void traverse(G, x) {
    visit(x);
    x.visited = true;
    for (each vertex w adjacent to v) {
        if (!w.visited))
            traverse(G, w);
    }
}
```
Example of Recursive DFS

What is the order vertices are visited?

1
2
3
4
5
6
7
8
Example

The vertices are visited in the order 1, 2, 5, 6, 3, 4, 7 and 8. This is different than the order we got when using a stack!
Complexity of DFS

- DFS as implemented above (with adjacency lists) on a graph with \( e \) edges and \( n \) vertices has complexity \( O(n + e) \).

- To see why observe that on no vertex is `traverse` called more than once, because as soon as we call `traverse` with parameter \( x \), we mark \( x \) visited and we never call `traverse` on a vertex that has previously been marked as visited.

- Thus, the total time spent going down the adjacency lists is proportional to the lengths of those lists, that is \( O(e) \)

- The initialization steps in `graph_dfs` have complexity \( O(n) \)

- Thus, the total complexity is \( O(n + e) \)
Complexity of DFS

• If DFS is implemented using an adjacency matrix, then its complexity will be \(O(n^2)\).

• If the graph is **dense (πυκνός)**, that is, it has close to \(O(n^2)\) edges the difference of the two implementations is minor as they would both run in \(O(n^2)\) time.

• If the graph is **sparse (αραιός)**, that is, it has close to \(O(n)\) edges, then the adjacency matrix approach would be much slower than the adjacency list approach.
Complexity of BFS

• BFS with adjacency lists has the same complexity as DFS i.e., $O(n + e)$. 
Readings

• T. A. Standish. *Data Structures, Algorithms and Software Principles in C*. Chapter 10


