

# Multi-Way Search Trees

Κ08 Δομές Δεδομένων και Τεχνικές Προγραμματισμού

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# Motivation

- We keep the **ordering** idea of BSTs
  - **Fast search**, by excluding whole subtrees
- And add **more than two children** for each node
  - Gives more flexibility in restructuring the tree
  - And new ways to **keep it balanced**

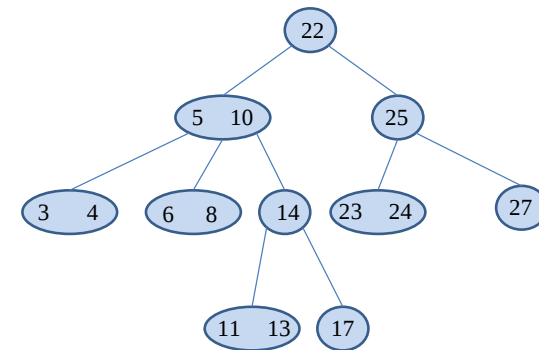
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# Multi-way search trees

- $d$ -node: a node with  $d$  children
- Each **internal**  $d$ -node stores  $d - 1$  **ordered** values  $k_1 < \dots < k_{d-1}$ 
  - **No duplicate** values in the whole tree
- All values in a **subtree** lie **in-between** the corresponding node values
  - For all values  $l$  in the  $i$ -th subtree:  $k_{i-1} < l < k_i$
  - Convention:  $k_0 = -\infty, k_d = +\infty$
- $m$ -way search tree: all nodes have **at most**  $m$  children
  - A BST is a 2-way search tree

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# Example multi-way search tree



$m = 3$

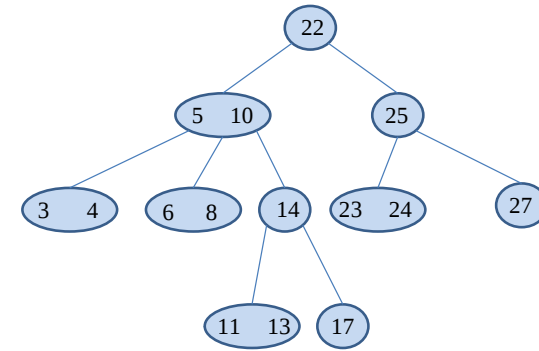
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## Searching in a multi-way search tree

- Simple adaptation of the algorithm for BSTs
- Start from the root, traverse towards the leaves
- In each node, there is a **single subtree** that can possibly contain a value  $l$ 
  - The subtree  $i$  such that  $k_{i-1} < l < k_i$
  - Continue in that subtree

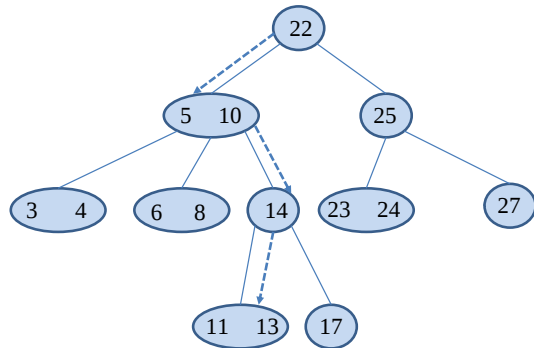
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## Example multi-way search tree



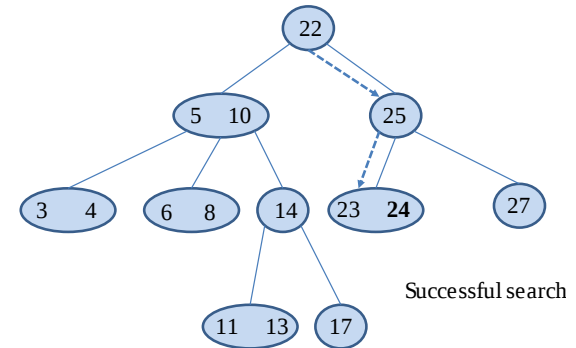
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## Search for value 12



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## Search for value 24



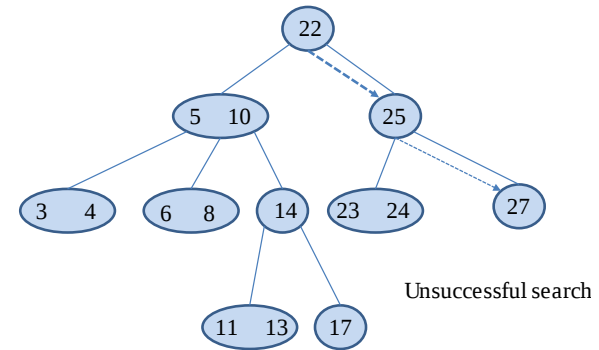
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## Insertion in a multi-way search tree

- Again, simple adaptation of BSTs
  - **But:** we don't always need to create a new node
  - We can insert in an existing one if there is space
- Start with a search for the value  $l$  we want to insert
- If found, stop (no duplicates)
- If not found, insert at the **leaf** we reached
  - If full, create an  $i$ -th child, such that  $k_{i-1} < l < k_i$

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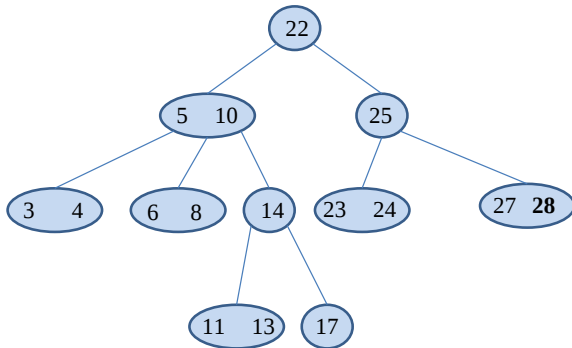
## Insert value 28



$m = 3$

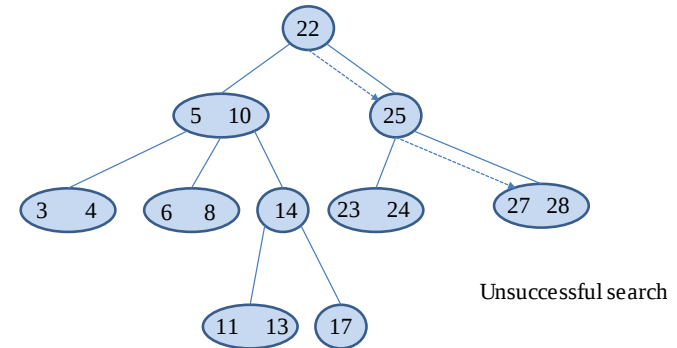
10

## Value 28 inserted



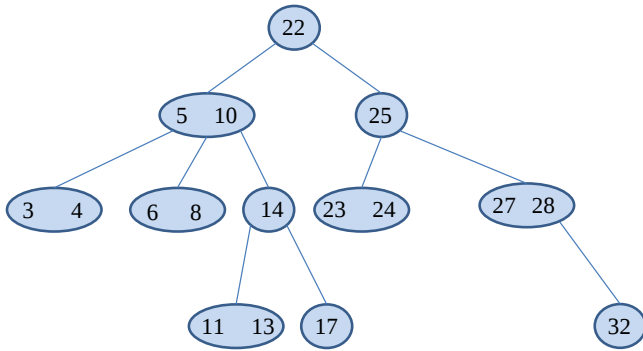
11

## Insert value 32



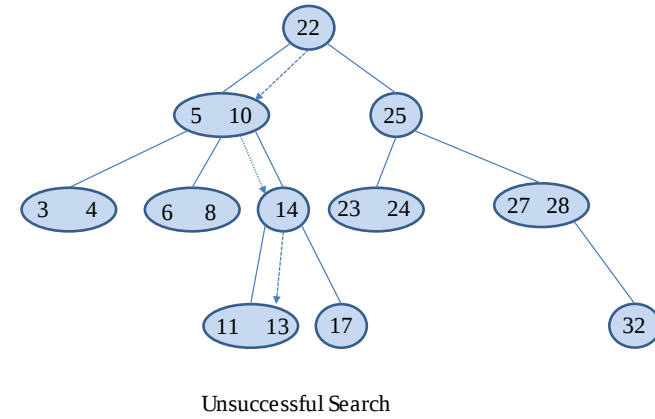
12

## Value 32 inserted



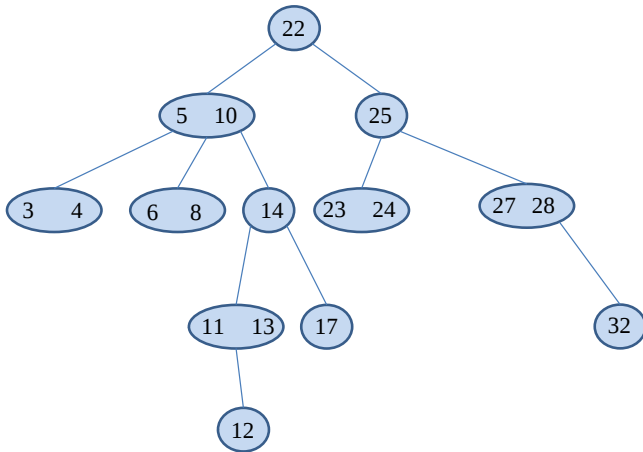
13

## Insert value 12



14

## Value 12 inserted



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## Deletion from a multi-way search tree

Left as an exercise.

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## Complexity of operations

- We need to traverse the tree from the root to a leaf
- The time spent at each node is constant
  - Eg. find  $i$  such that  $k_{i-1} < l < k_i$
  - Assuming  $m$  is **fixed!**
- So as usual all complexities are  $O(h)$ 
  - $O(n)$  in the worst-case

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## Balanced multi-way search trees

- Similarly to BSTs we need to keep the tree **balanced**
  - So that  $h = O(\log n)$
- AVL where a kind of balanced BSTs
- We will study two kinds of **balanced multi-way** search trees:
  - **2-3 trees**
  - **2-3-4 trees** (also known as (2,4) trees)

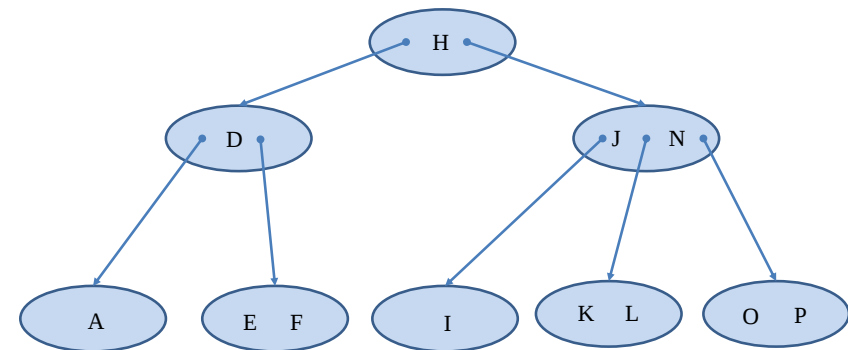
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## 2-3 trees

- A **2-3 tree** is a 3-way search tree which has the following properties
- **Size property**
  - Each node contains **1 or 2 values**  
(so each **internal** node contains **2 or 3 children**)
- **Depth property**
  - All **leaves** have the **same depth** (lie on the same level)

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## Example of 2-3 tree



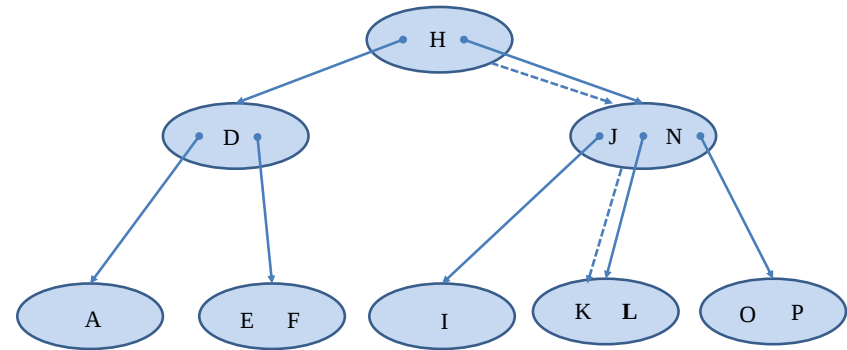
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## Height of 2-3 trees

- **All nodes** at **all levels** except the last one are **internal**
  - And each internal node has at least 2 children
  - So at level  $i$  we have at least  $2^i$  nodes
- Hence  $n \geq 2^h$ , in other words  $h = O(\log n)$
- So we can search for an element in time  $O(\log n)$ 
  - Using the standard algorithm for  $m$ -way trees

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## Search for L



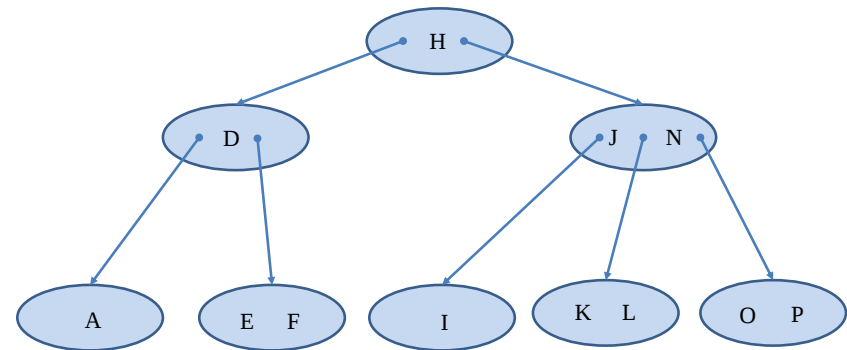
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## Insertion in 2-3-trees

- We can start by following the generic algorithm for  $m$ -way trees
- Search for the value  $l$  we want to insert
- If found, stop (no duplicates)
- If not found, insert at the **leaf** we reached

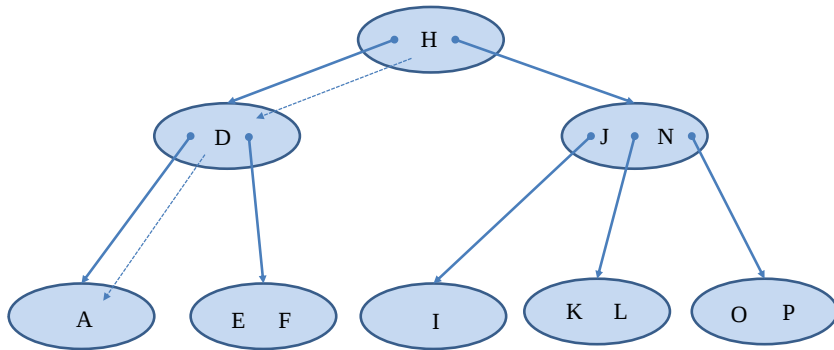
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## Example: insert B



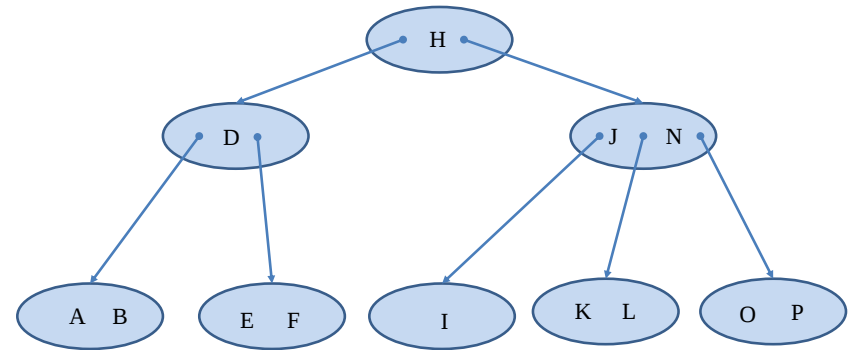
24

## Example: insert B



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## Example: result



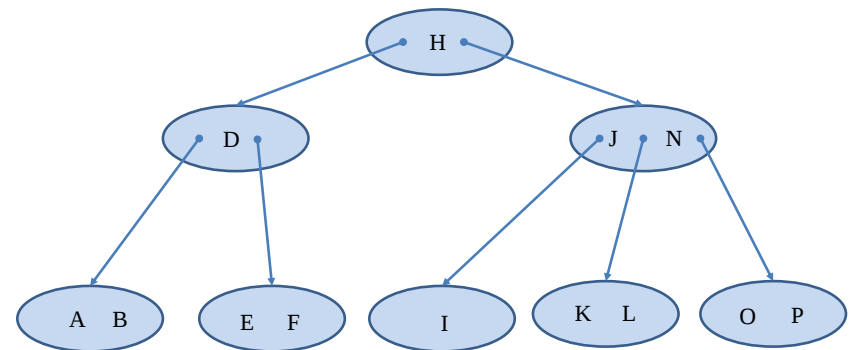
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## Insertion in 2-3-trees

- But what if there is **no space at the leaf** (overflow)?
- The standard algorithm will insert a child at the leaf
  - But this **violates the depth property!**
  - The new leaf is not at the same level
- Different strategy
  - **split** the overflowed node into two nodes
  - pass the **middle value** to the parent (**separator** of the two nodes)
- The middle value might **overflow the parent**
  - Same procedure: split and send the middle value up

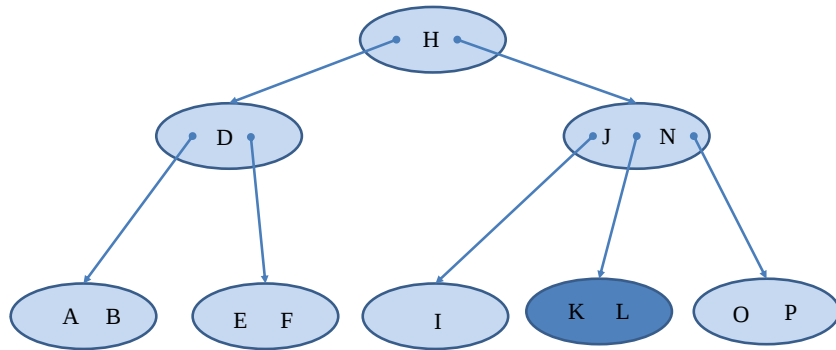
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## Example: insert M



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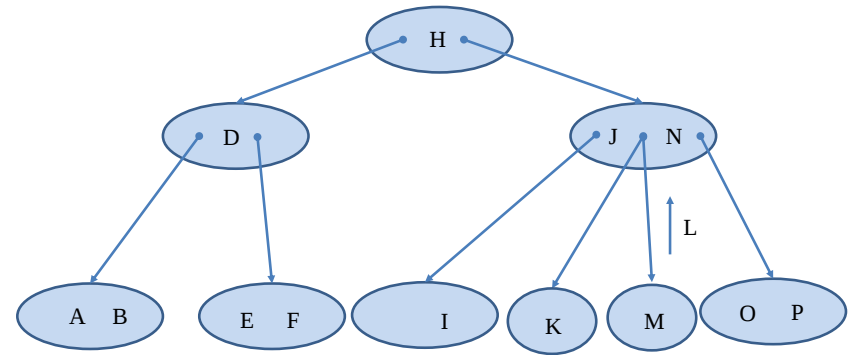
## Example: insert M



M overflows this node.

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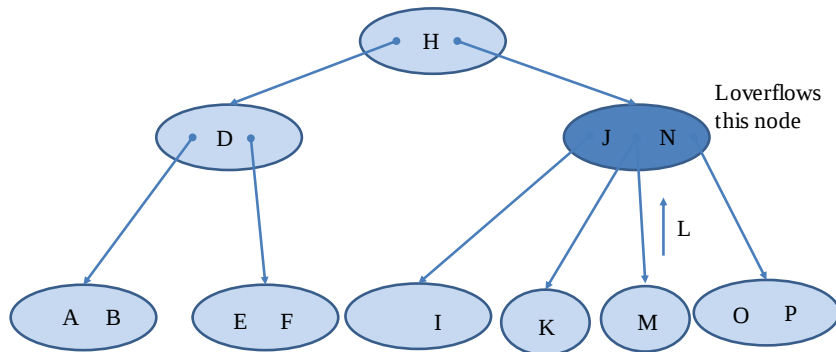
## Example: insert M



The node is split in two and L is passed to the parent node

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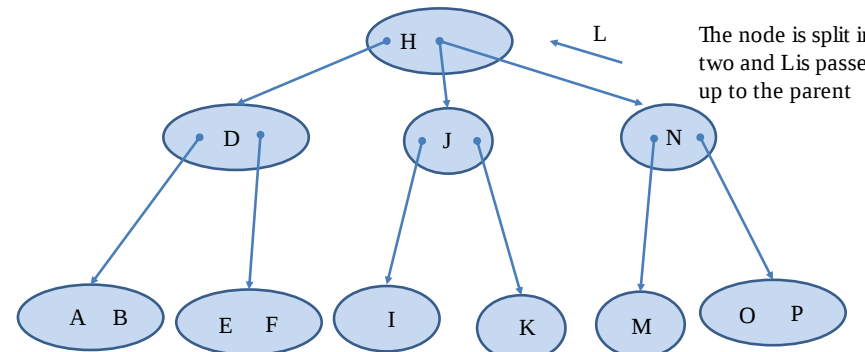
## Example: insert M



L overflows this node

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## Example: insert M

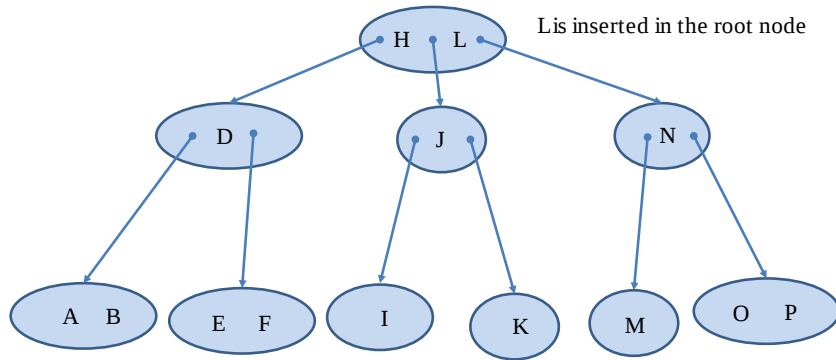


The node is split in two and L is passed up to the parent

32

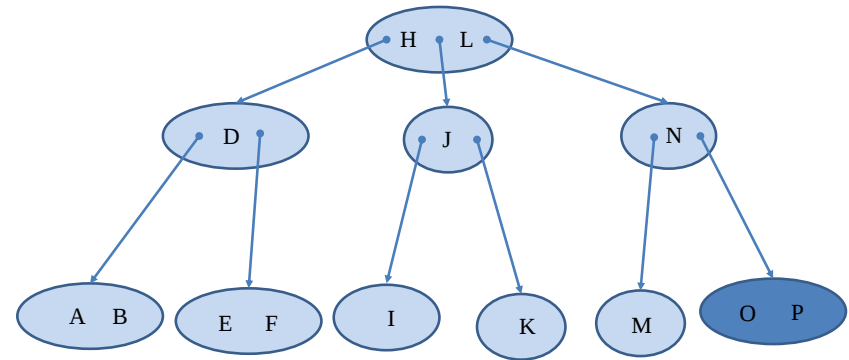


## Example: result



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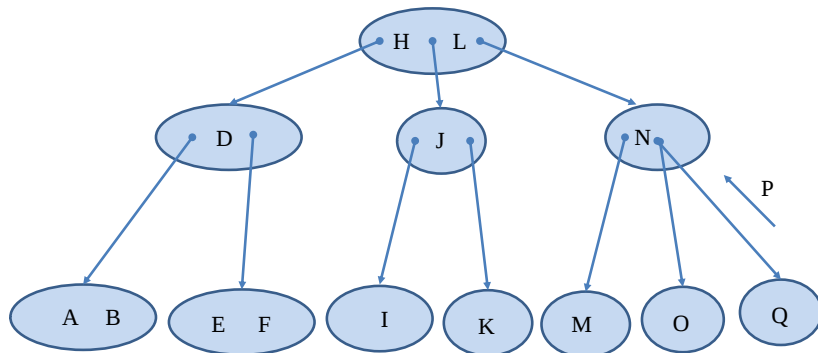
## Example: insert Q



Q overflows  
this node

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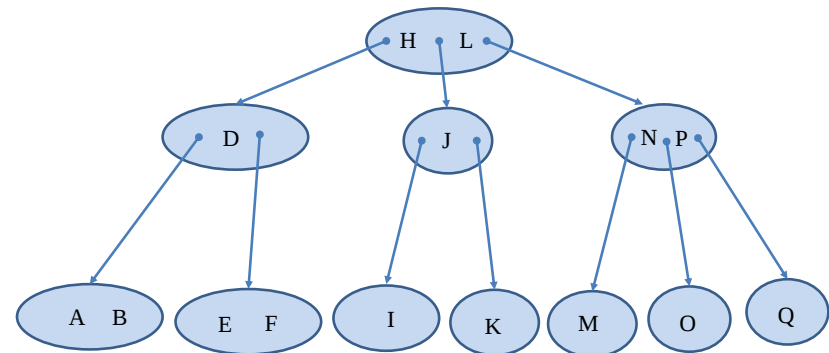
## Example: insert Q



This node is split up  
and P is passed up

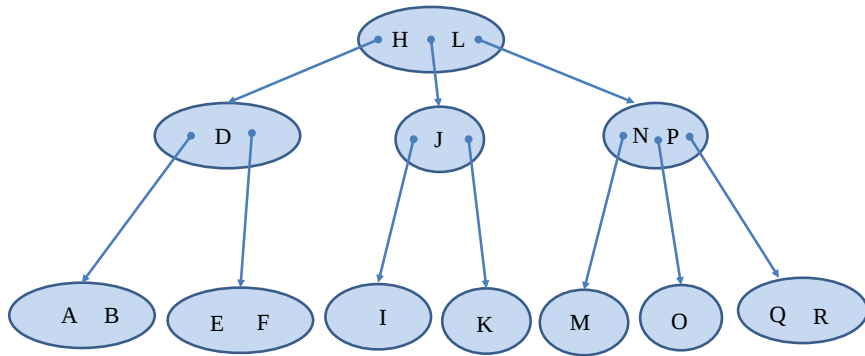
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## Example: result



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## Example: insert R



R is inserted in the node with Q where there is space.

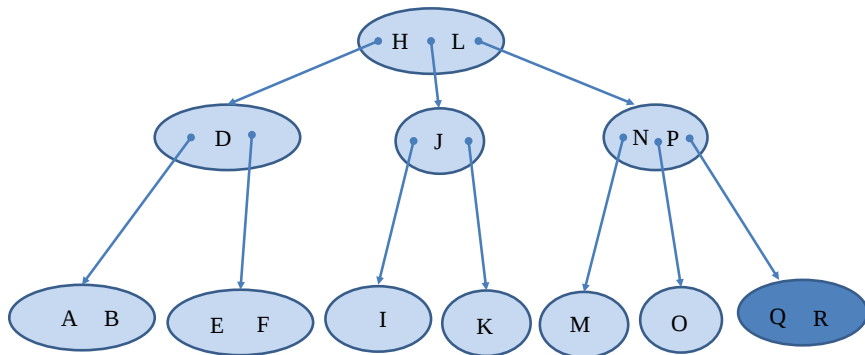
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## Insertion in 2-3-trees

- The **root** might also **overflow**
- Same procedure
  - Split it
  - The middle value moves up, creating a **new root**
- This is the **only** operation that **increases** the tree's **height**
  - It increases the depth of **all nodes** simultaneously
  - 2-3-trees grow at the root, not at the leaves!

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## Example: insert S

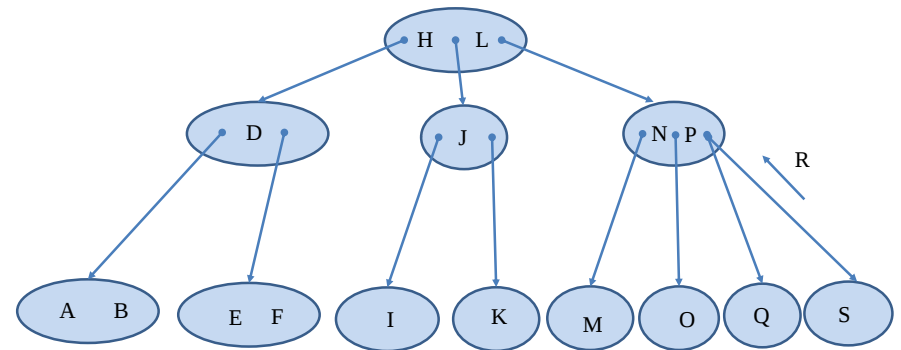


S overflows  
this node

S overflows this node

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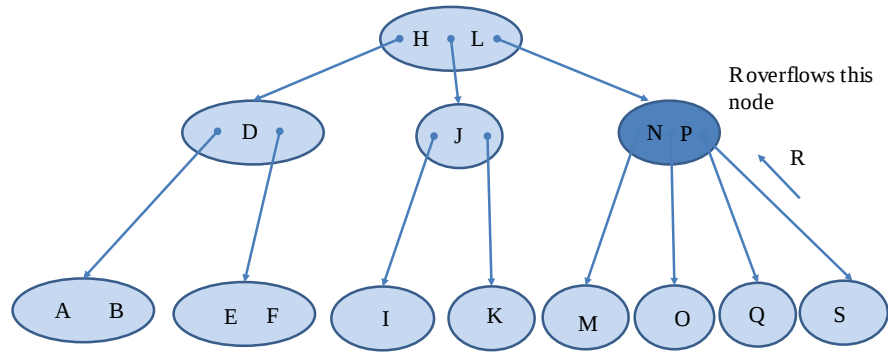
## Example: insert S



This node is split  
and R is sent up

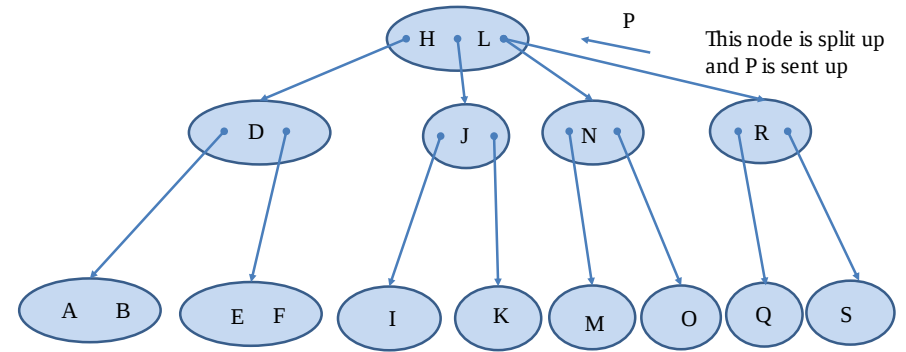
40

## Example: insert S



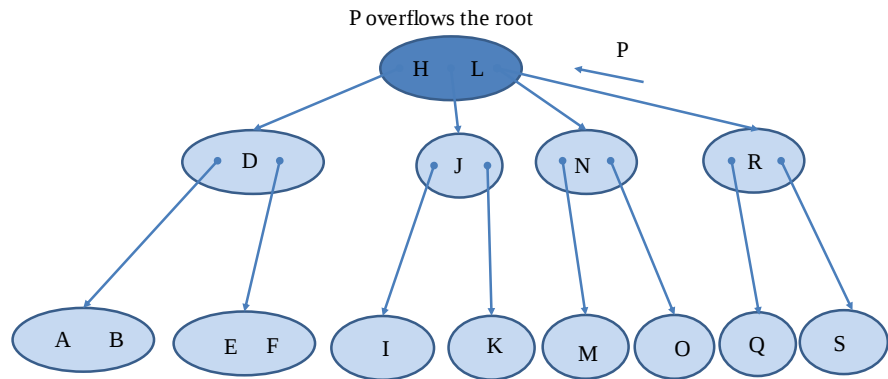
41

## Example: insert S



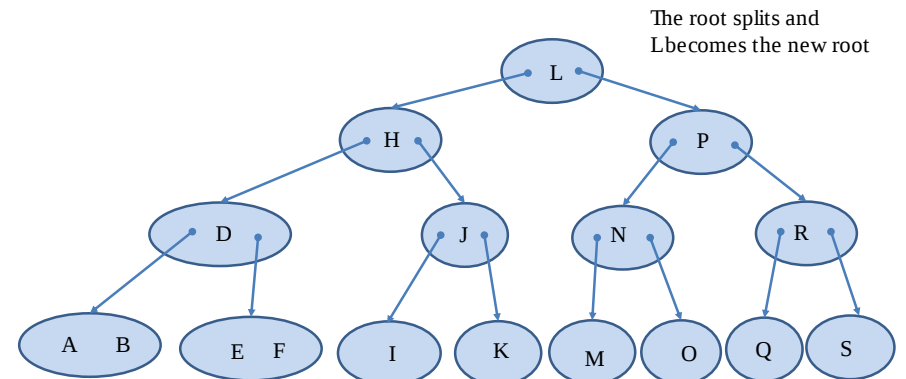
42

## Example: insert S



43

## Example: result



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## Complexity of insertion

- We traverse the tree
  - From the root to a leaf when searching
  - From the leaf back to the root while splitting
- Each split takes constant time
  - We do at most  $h + 1$  of them
- So in total  $O(h) = O(\log n)$  steps
  - Recall, the tree is balanced

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## (2,4) trees

- A **(2,4) tree** (or 2-3-4 tree) is a 4-way search tree with 2 extra properties
- **Size property**
  - Each node contains between **1 and 3 values**  
(so each **internal** node contains between **2 and 4 children**)
- **Depth property**
  - All **leaves** have the **same depth** (lie on the same level)
- Such trees are **balanced**
  - $h = O(\log n)$
  - Proof: exercise

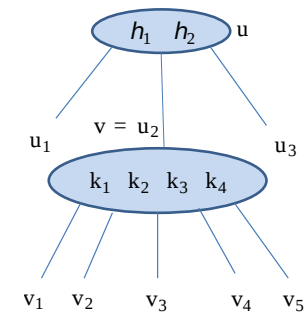
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## Insertion in (2,4) trees

- Same as for 2-3-trees
  - Search for the value
  - Insert at a leaf
- In case of an overflow (5-node)
  - Split it into a 3-node and a 2-node
  - Move the separator value  $k_3$  to the parent

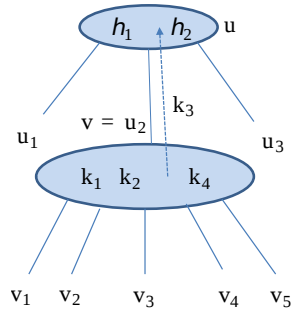
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## Overflow at a 5-node



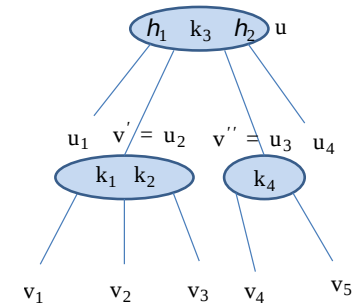
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## The separating value is sent to the parent node



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## Node replaced with a 3-node and a 2-node



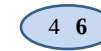
50

## Example: insert 4



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## Example: insert 6



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## Example: insert 12

4 6 12

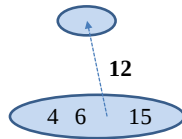
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## Example: insert 15 - overflow

4 6 12 15

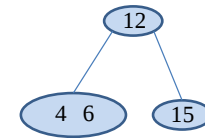
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## Creation of new root node



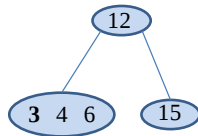
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## Split



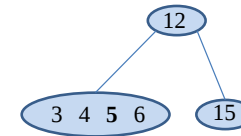
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## Example: insert 3



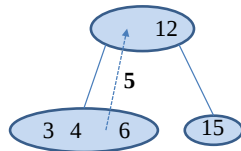
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## Example: insert 5 - overflow



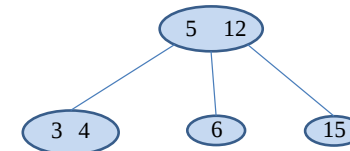
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## 5 is sent to the parent node



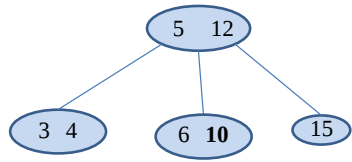
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## Split



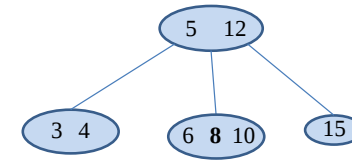
60

## Example: insert 10



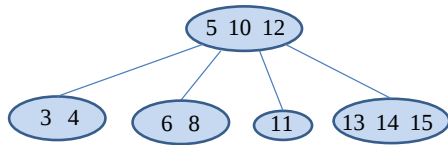
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## Example: insert 8



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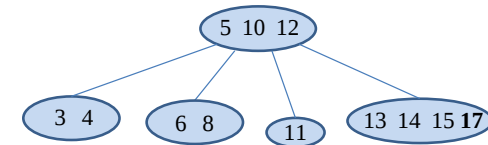
## Example



Inserted 11, 13 and 14.

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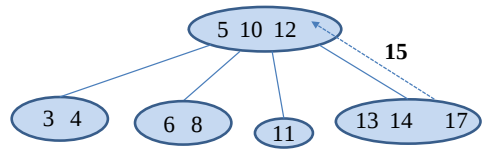
## Example: insert 17 - overflow



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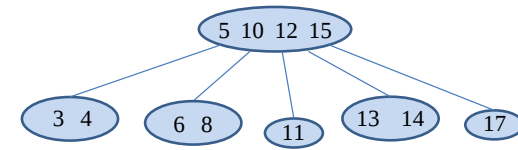


## Split and send 15 to the parent node



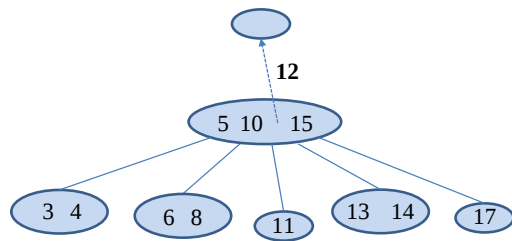
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## The root overflows



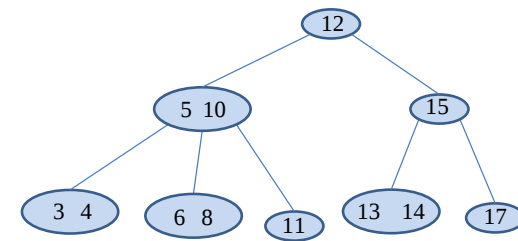
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## Creation of new root



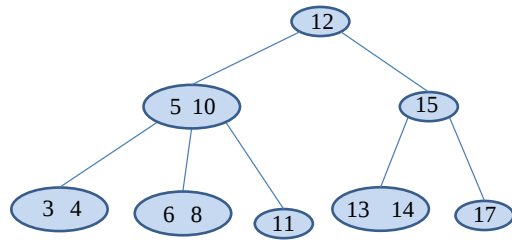
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## Split



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## Final tree



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## Complexity

- Same as for 2-3-trees
  - At most  $h$  splits
  - Each split is constant time
- $O(\log n)$ 
  - Because the tree is balanced

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## Removal in (2,4) trees

- To remove a value  $k_i$  from an **internal** node
  - Replace with its **predecessor** (or its **successor**)
  - Right-most value in the  $i$ -th subtree
  - Similar to the BST case of nodes with two children
- To remove a value from a **leaf**
  - We simply remove it
  - But it might violate the **size** property (**underflow**)

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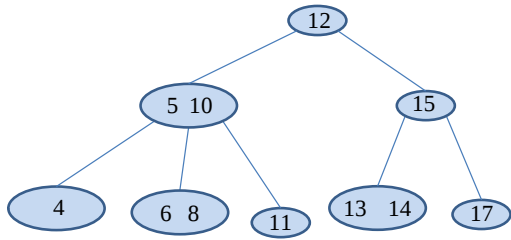
## Fixing underflows

Two strategies for fixing an underflow at  $\nu$

- Is there an **immediate sibling**  $w$  with a “spare” value? (2 or 3 values)
- If so, we do a **transfer** operation
  - Move a value of  $w$  to its parent  $u$
  - Move a value of the parent  $u$  to  $\nu$
- If not, we do a **fusion** operation
  - Merge  $\nu$  and  $w$ , creating a new node  $\nu'$
  - Move a value from the parent  $u$  to  $\nu'$
  - This might **underflow the parent**, continue the same procedure there

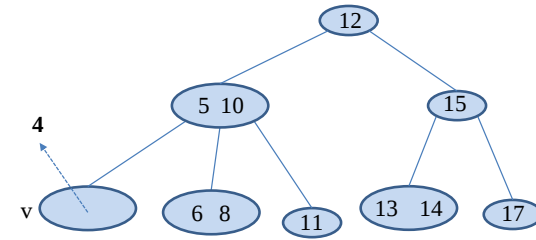
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## Initial tree



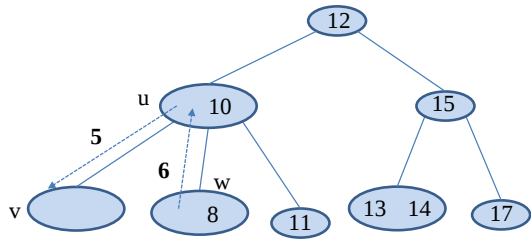
73

## Remove 4



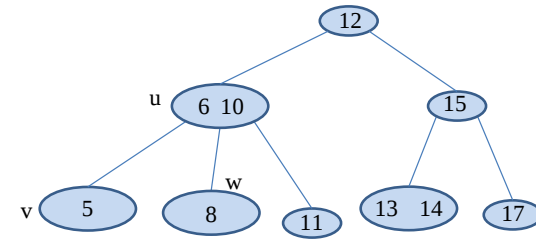
74

## Transfer



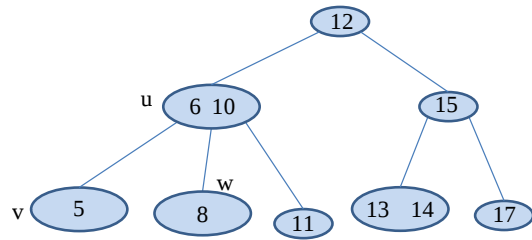
75

## After the transfer



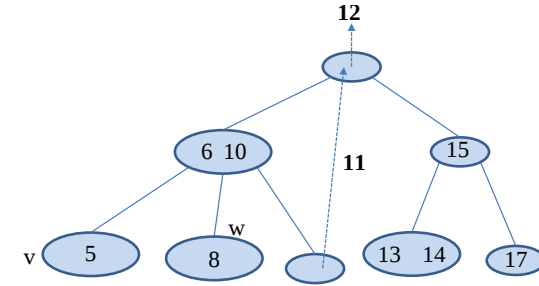
76

## Remove 12



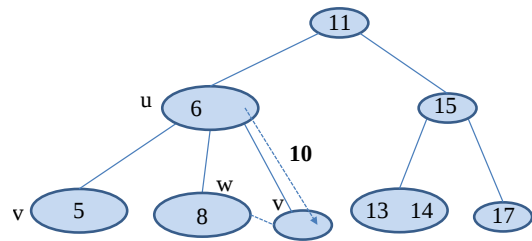
77

## Remove 12



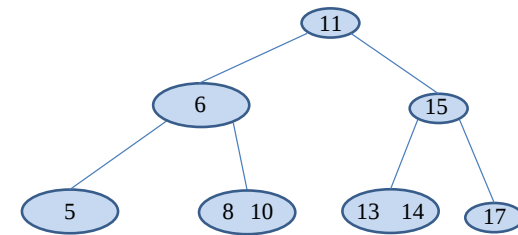
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## Fusion of and



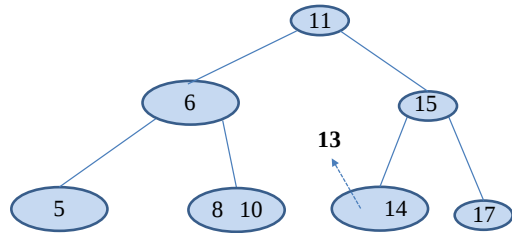
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## After the fusion



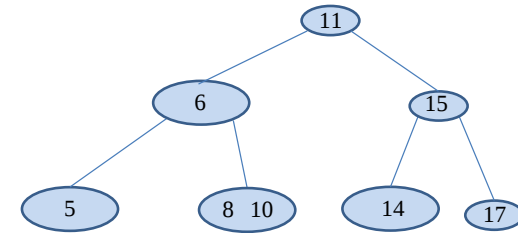
80

## Remove 13



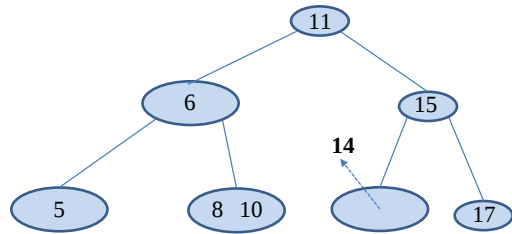
81

## After the removal of 13



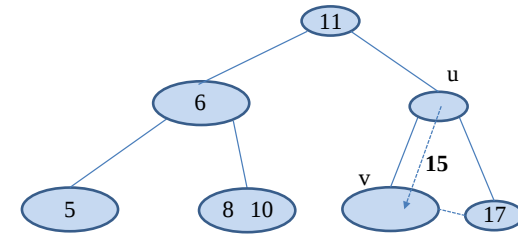
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## Remove 14 - underflow



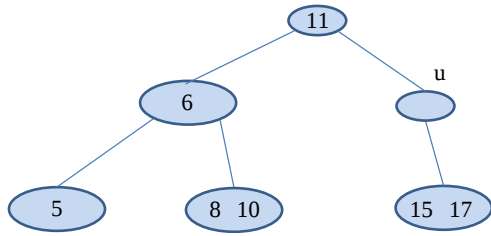
83

## Fusion



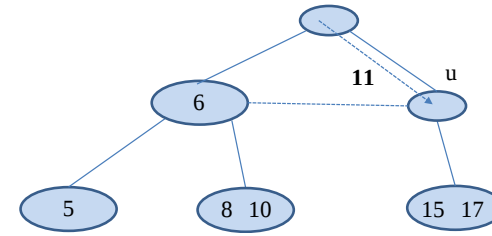
84

## Underflow at



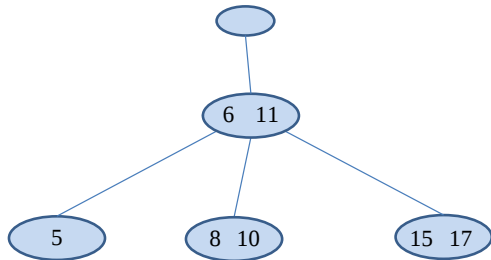
85

## Fusion



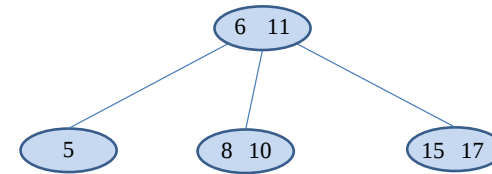
86

## Remove the root



87

## Final tree



88

## Readings

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