Multi-Way Search Trees

Motivation

- We keep the **ordering** idea of BSTs
  - Fast search, by excluding whole subtrees
- And add **more than two children** for each node
  - Gives more flexibility in restructuring the tree
  - And new ways to **keep it balanced**

Multi-way search trees

- $d$-node: a node with $d$ children
- Each internal $d$-node stores $d - 1$ ordered values $k_1 < \ldots < k_{d-1}
  - No duplicate values in the whole tree
- All values in a **subtree** lie in-between the corresponding node values
  - For all values $l$ in the $i$-th subtree: $k_{i-1} < l < k_i$
  - Convention: $k_0 = -\infty$, $k_d = +\infty$
- $m$-way search tree: all nodes have **at most** $m$ children
  - A BST is a 2-way search tree

Example multi-way search tree

\[
\begin{array}{c}
22 \\
5 10 25 \\
3 4 6 8 14 23 24 27 \\
11 13 17 \\
\end{array}
\]

$m = 3$
Searching in a multi-way search tree

- Simple adaptation of the algorithm for BSTs
- Start from the root, traverse towards the leaves
- In each node, there is a single subtree that can possibly contain a value $l$
  - The subtree $i$ such that $k_{i-1} < l < k_i$
  - Continue in that subtree

Example multi-way search tree

Search for value 12

Search for value 24

Successful search

Unsuccessful search
Insertion in a multi-way search tree

- Again, simple adaptation of BSTs
  - But: we don’t always need to create a new node
  - We can insert in an existing one if there is space
- Start with a search for the value $l$ we want to insert
  - If found, stop (no duplicates)
  - If not found, insert at the leaf we reached
    - If full, create an $i$-th child, such that $k_{i-1} < l < k_i$

Insert value 28

Unsuccessful search

$m = 3$

Value 28 inserted

Insert value 32

Unsuccessful search
Value 32 inserted

Insert value 12

Value 12 inserted

Deletion from a multi-way search tree

Left as an exercise.
**Complexity of operations**

- We need to traverse the tree from the root to a leaf
- The time spent at each node is constant
  - Eg. find $i$ such that $k_{i-1} < l < k_i$
  - Assuming $m$ is fixed!
- So as usual all complexities are $O(n)$
  - $O(h)$ in the worst-case

**Balanced multi-way search trees**

- Similarly to BSTs we need to keep the tree balanced
  - So that $h = O(\log n)$
- AVL where a kind of balanced BSTs
- We will study two kinds of balanced multi-way search trees:
  - 2-3 trees
  - 2-3-4 trees (also known as (2,4) trees)

**2-3 trees**

- A 2-3 tree is a 3-way search tree which has the following properties
  - Size property
    - Each node contains 1 or 2 values (so each internal node contains 2 or 3 children)
  - Depth property
    - All leaves have the same depth (lie on the same level)

**Example of 2-3 tree**
**Height of 2-3 trees**

- All nodes at all levels except the last one are **internal**
  - And each internal node has at least 2 children
  - So at level $i$ we have at least $2^i$ nodes
- Hence $n \geq 2^h$, in other words $h = O(\log n)$
- So we can search for an element in time $O(\log n)$
  - Using the standard algorithm for $m$-way trees

**Insertion in 2-3-trees**

- We can start by following the generic algorithm for $m$-way trees
- Search for the value $l$ we want to insert
- If found, stop (no duplicates)
- If not found, insert at the leaf we reached

**Search for L**

**Example: insert B**
**Example: insert B**

- But what if there is **no space at the leaf** (overflow)?
- The standard algorithm will insert a child at the leaf
  - But this **violates the depth property**!
  - The new leaf is not at the same level
- Different strategy
  - **split** the overflowed node into two nodes
  - pass the **middle value** to the parent (**separator** of the two nodes)
- The middle value might **overflow the parent**
  - Same procedure: split and send the middle value up

**Example: result**

**Insertion in 2-3-trees**

**Example: insert M**
Example: insert M

M overflows this node.

The node is split in two and L is passed to the parent node.

L overflows this node.

The node is split in two and L is passed up to the parent.
Example: result

L is inserted in the root node

Example: insert Q

Q overflows this node

Example: insert Q

This node is split up and P is passed up

Example: result
Example: insert R

![Diagram of 2-3 tree with insertion of R]

R is inserted in the node with Q where there is space.

Insertion in 2-3-trees

- The root might also overflow
- Same procedure
  - Split it
  - The middle value moves up, creating a new root
- This is the only operation that increases the tree’s height
  - It increases the depth of all nodes simultaneously
  - 2-3-trees grow at the root, not at the leaves!

Example: insert S

![Diagram of 2-3 tree with insertion of S]

S overflows this node

S overflows this node

Example: insert S

![Diagram of 2-3 tree with insertion of S]

This node is split and R is sent up

This node is split and R is sent up
Example: insert S

- H + L
- D J N P
- R
- A B E F I K M O Q S

Roverflows this node

Example: insert S

- H + L
- D J N P
- R
- A B E F I K M O Q S

This node is split up and P is sent up

Example: insert S

- H + L
- D J N P
- R
- A B E F I K M O Q S

P overflows the root

Example: result

- H + L
- D J N P
- R
- A B E F I K M O Q S

The root splits and L becomes the new root
Complexity of insertion

- We traverse the tree  
  - From the root to a leaf when searching  
  - From the leaf back to the root while splitting  
- Each split takes constant time  
  - We do at most $h + 1$ of them  
- So in total $O(h) = O(\log n)$ steps  
  - Recall, the tree is balanced

($2,4$) trees

- A ($2,4$) tree (or 2-3-4 tree) is a 4-way search tree with 2 extra properties
  - Size property  
    - Each node contains between 1 and 3 values  
      (so each internal node contains between 2 and 4 children)
  - Depth property  
    - All leaves have the same depth (lie on the same level)
- Such trees are balanced  
  - $h = O(\log n)$  
  - Proof: exercise

Insertion in ($2,4$) trees

- Same as for 2-3-trees  
  - Search for the value  
  - Insert at a leaf  
- In case of an overflow (5-node)  
  - Split it into a 3-node and a 2-node  
  - Move the separator value $k_3$ to the parent

Overflow at a 5-node
The separating value is sent to the parent node

Node replaced with a 3-node and a 2-node

Example: insert 4

Example: insert 6
Example: insert 12

4 6 12

Example: insert 15 - overflow

4 6 12 15

Creation of new root node

4 6 15

Split

12

4 6 15
Example: insert 3

Example: insert 5 - overflow

5 is sent to the parent node

Split
Example: insert 10

Example: insert 8

Example

Inserted 11, 13 and 14.

Example: insert 17 - overflow
Split and send 15 to the parent node

The root overflows

Creation of new root

Split
**Final tree**

![Tree Diagram]

**Complexity**
- Same as for 2-3-trees
  - At most $h$ splits
  - Each split is constant time
- $O(\log n)$
  - Because the tree is balanced

**Removal in (2,4) trees**
- To remove a value $k_i$ from an **internal** node
  - Replace with its **predecessor** (or its **successor**)
  - Right-most value in the $i$-th subtree
  - Similar to the BST case of nodes with two children
- To remove a value from a **leaf**
  - We simply remove it
  - But it might violate the **size** property (**underflow**)

**Fixing underflows**
Two strategies for fixing an underflow at $\nu$
- Is there an **immediate sibling** $w$ with a “spare” value? (2 or 3 values)
- If so, we do a **transfer** operation
  - Move a value of $w$ to its parent $u$
  - Move a value of the parent $u$ to $\nu$
- If not, we do a **fusion** operation
  - Merge $\nu$ and $w$, creating a new node $\nu'$
  - Move a value from the parent $u$ to $\nu'$
  - This might **underflow the parent**, continue the same procedure there
Remove 12

Fusion of and

After the fusion
Remove 13

After the removal of 13

Remove 14 - underflow

Fusion
Underflow at

Fusion

Remove the root

Final tree
Readings

- T. A. Standish. *Data Structures, Algorithms and Software Principles in C*. Section 9.9
- R. Sedgewick. *Αλγόριθμοι σε C*. 3η Αμερικανική Έκδοση. Εκδόσεις Κλειδάριθμος. Section 13.3