Multi-Way Search Trees

Motivation
- We keep the ordering idea of BSTs
  - Fast search, by excluding whole subtrees
- And add more than two children for each node
  - Gives more flexibility in restructuring the tree
  - And new ways to keep it balanced

Multi-way search trees
- $d$-node: a node with $d$ children
- Each internal $d$-node stores $d - 1$ ordered values $k_1 < \ldots < k_{d-1}$
  - No duplicate values in the whole tree
- All values in a subtree lie in-between the corresponding node values
  - For all values $l$ in the $i$-th subtree: $k_{i-1} < l < k_i$
  - Convention: $k_0 = -\infty$, $k_d = +\infty$
- $m$-way search tree: all nodes have at most $m$ children
  - A BST is a 2-way search tree

Example multi-way search tree

$m = 3$
Searching in a multi-way search tree

- Simple adaptation of the algorithm for BSTs
- Start from the root, traverse towards the leaves
- In each node, there is a **single subtree** that can possibly contain a value $l$
  - The subtree $i$ such that $k_{i-1} < l < k_i$
  - Continue in that subtree

Example multi-way search tree

Search for value 12

Unsuccessful search

Search for value 24

Successful search
Insertion in a multi-way search tree

- Again, simple adaptation of BSTs
  - **But**: we don’t always need to create a new node
  - We can insert in an existing one if there is space
- Start with a search for the value \( l \) we want to insert
- If found, stop (no duplicates)
- If not found, insert at the leaf we reached
  - If full, create an \( i \)-th child, such that \( k_{i-1} < l < k_i \)

**Insert value 28**

Value 28 inserted

**Insert value 32**

Value 28 inserted
Value 32 inserted

Value 12 inserted

Insert value 12

Unsuccessful Search

Deletion from a multi-way search tree

Left as an exercise.
Complexity of operations

- We need to traverse the tree from the root to a leaf
- The time spent at each node is constant
  - Eg. find $i$ such that $k_{i-1} < l < k_i$
  - Assuming $m$ is fixed!
- So as usual all complexities are $O(h)$
  - $O(n)$ in the worst-case

Balanced multi-way search trees

- Similarly to BSTs we need to keep the tree balanced
  - So that $h = O(\log n)$
- AVL where a kind of balanced BSTs
- We will study two kinds of balanced multi-way search trees:
  - 2-3 trees
  - 2-3-4 trees (also known as 2-4 trees)

2-3 trees

- A 2-3 tree is a 3-way search tree which has the following properties
  - Size property
    - Each node contains 1 or 2 values
    - Internal nodes with $n$ values have exactly $n + 1$ children
  - Depth property
    - All leaves have the same depth (lie on the same level)

Example of 2-3 tree
**Height of 2-3 trees**

- **All nodes at all levels** except the last one are **internal**
  - And each internal node has at least 2 children
  - So at level $i$ we have at least $2^i$ nodes
- Hence $n \geq 2^h$, in other words $h = O(\log n)$
- So we can search for an element in time $O(\log n)$
  - Using the standard algorithm for $m$-way trees

**Insertion in 2-3-trees**

- We can start by following the generic algorithm for $m$-way trees
  - Search for the value $l$ we want to insert
    - If found, stop (no duplicates)
    - If not found, insert at the **leaf** we reached

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**Search for L**

- Diagram showing a 2-3 tree with nodes labeled A, B, C, D, E, F, I, J, K, L, N, O, P.
- Illustrates the process of searching for a value in a 2-3 tree.

**Example: insert B**

- Diagram showing a 2-3 tree with nodes labeled A, B, C, D, E, F, I, J, K, L, N, O, P.
- Illustrates the process of inserting a value into a 2-3 tree.
Insertion in 2-3-trees

- But what if there is no space at the leaf (overflow)?
- The standard algorithm will insert a child at the leaf
  - But this violates the depth property!
  - The new leaf is not at the same level
- Different strategy
  - split the overflowed node into two nodes
  - pass the middle value to the parent (separator of the two nodes)
- The middle value might overflow the parent
  - Same procedure: split and send the middle value up
Example: insert M

M overflows this node.

The node is split in two and L is passed to the parent node.

The node is split in two and L is passed up to the parent.
Example: result

L is inserted in the root node

Example: insert Q

Q overflows this node

Example: insert Q

This node is split up and P is passed up

Example: result
**Example: insert R**

R is inserted in the node with Q where there is space.

**Insertion in 2-3-trees**

- The root might also **overflow**
- Same procedure
  - Split it
  - The middle value moves up, creating a new root
- This is the only operation that **increases** the tree’s **height**
  - It increases the depth of all nodes simultaneously
  - 2-3-trees grow at the root, not at the leaves!

**Example: insert S**

S overflows this node

This node is split and R is sent up
Example: insert S

H + L
D
J
N P
R
A B E F I K M O Q S

Example: insert S

H + L
D
J
N
R
A B E F I K M O Q S

Example: insert S

H + L
D
J
N
R
P
A B E F I K M O Q S

Example: result

The root splits and L becomes the new root

L
H
P
D
J
N
R
A B E F I K M O Q S
**Complexity of insertion**

- We traverse the tree
  - From the root to a leaf when searching
  - From the leaf back to the root while splitting
- Each split takes constant time
  - We do at most $h + 1$ of them
- So in total $O(h) = O(\log n)$ steps
  - Recall, the tree is balanced

**2-4 trees**

- A 2-4 tree (or 2-3-4 tree) is a 4-way search tree with 2 extra properties
- **Size property**
  - Each node contains between 1 and 3 values
  - Internal nodes with $n$ values have exactly $n + 1$ children
- **Depth property**
  - All leaves have the same depth (lie on the same level)
- Such trees are balanced
  - $h = O(\log n)$
  - Proof: exercise

**Insertion in 2-4 trees**

- Same as for 2-3-trees
  - Search for the value
  - Insert at a leaf
- In case of an overflow (5-node)
  - Split it into a 3-node and a 2-node
  - Move the separator value $k_3$ to the parent

**Overflow at a 5-node**

- $v_1, v_2, v_3, v_4, v_5$ are the children of the 5-node.
- $k_1, k_2, k_3, k_4$ are the values in the 5-node.
- $u_1, u_2, u_3, u$ are the parents of the children.
- The separator value $k_3$ is moved to the parent.
The separating value is sent to the parent node

Node replaced with a 3-node and a 2-node

Example: insert 4

Example: insert 6
Example: insert 12

Example: insert 15 - overflow

Creation of new root node

Split
Example: insert 3

Example: insert 5 - overflow

5 is sent to the parent node

Split
Example: insert 10

Example: insert 8

Example

Example: insert 17 - overflow

Inserted 11, 13 and 14.
Split and send 15 to the parent node

The root overflows

Creation of new root

Split
Final tree

3 4 6 8 11 13 14 17
12
5 10 15
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Complexity
- Same as for 2-3-trees
  - At most $h$ splits
  - Each split is constant time
- $O(\log n)$
  - Because the tree is balanced

Removal in 2-4 trees
- To remove a value $k_i$ from an internal node
  - Replace with its predecessor (or its successor)
  - Right-most value in the $i$-th subtree
  - Similar to the BST case of nodes with two children
- To remove a value from a leaf
  - We simply remove it
  - But it might violate the size property (underflow)

Fixing underflows
Two strategies for fixing an underflow at $\nu$
- Is there an immediate sibling $w$ with a “spare” value? (2 or 3 values)
  - If so, we do a transfer operation
    - Move a value of $w$ to its parent $u$
    - Move a value of the parent $u$ to $\nu$
  - If not, we do a fusion operation
    - Merge $\nu$ and $w$, creating a new node $\nu'$
    - Move a value from the parent $u$ to $\nu'$
    - This might underflow the parent, continue the same procedure there
Initial tree

Remove 4

Transfer

After the transfer
After the fusion
Remove 13

After the removal of 13

Remove 14 - underflow

Fusion
Underflow at

Fusion

Remove the root

Final tree
Readings

- T. A. Standish. *Data Structures, Algorithms and Software Principles in C*. Section 9.9
- R. Sedgewick. *Αλγόριθμοι σε C*. 3η Αμερικανική Έκδοση. Εκδόσεις Κλειδάριθμος. Section 13.3