Multi-Way Search Trees

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Motivation

• We keep the **ordering** idea of BSTs
  - **Fast search**, by excluding whole subtrees

• And add **more than two children** for each node
  - Gives more flexibility in restructuring the tree
  - And news ways to **keep it balanced**
Multi-way search trees

- **$d$-node**: a node with $d$ children
- Each **internal** $d$-node stores $d - 1$ **ordered** values $k_1 < \ldots < k_{d-1}$
  - **No duplicate** values in the whole tree
- All values in a **subtree** lie **in-between** the corresponding node values
  - For all values $l$ in the $i$-th subtree: $k_{i-1} < l < k_i$
  - Convention: $k_0 = -\infty$, $k_d = +\infty$
- **$m$-way search tree**: all nodes have **at most** $m$ children
  - A BST is a 2-way search tree
Example multi-way search tree

$m = 3$
Searching in a multi-way search tree

• Simple adaptation of the algorithm for BSTs

• Start from the root, traverse towards the leaves

• In each node, there is **a single subtree** that can possibly contain a value $l$
  
  - The subtree $i$ such that $k_{i-1} < l < k_i$
  
  - Continue in that subtree
Example multi-way search tree
Search for value 12

Unsuccessful search
Search for value 24

Successful search
Insertion in a multi-way search tree

- Again, simple adaptation of BSTs
  - **But**: we don't always need to create a new node
  - We can insert in an existing one if there is space
- Start with a search for the value $l$ we want to insert
- If found, stop (no duplicates)
- If not found, insert at the **leaf** we reached
  - If full, create an $i$-th child, such that $k_{i-1} < l < k_i$
Insert value 28

Unsuccessful search

$m = 3$
Value 28 inserted
Insert value 32

Unsuccessful search
Value 32 inserted
Insert value 12

Unsuccessful Search
Value 12 inserted
Deletion from a multi-way search tree

Left as an exercise.
Complexity of operations

• We need to traverse the tree from the root to a leaf

• The time spent at each node is constant
  - Eg. find $i$ such that $k_{i-1} < l < k_i$
  - Assuming $m$ is fixed!

• So as usual all complexities are $O(h)$
  - $O(n)$ in the worst-case
Balanced multi-way search trees

• Similarly to BSTs we need to keep the tree balanced
  - So that $h = O(\log n)$

• AVL where a kind of balanced BSTs

• We will study two kinds of balanced multi-way search trees:
  - 2-3 trees
  - 2-3-4 trees (also known as 2-4 trees)
2-3 trees

• A 2-3 tree is a 3-way search tree which has the following properties

• Size property
  - Each node contains 1 or 2 values
    (so each internal node contains 2 or 3 children)

• Depth property
  - All leaves have the same depth (lie on the same level)
Example of 2-3 tree
Height of 2-3 trees

- **All nodes** at **all levels** except the last one are **internal**
  - And each internal node has at least 2 children
  - So at level $i$ we have at least $2^i$ nodes

- Hence $n \geq 2^h$, in other words $h = O(\log n)$

- So we can search for an element in time $O(\log n)$
  - Using the standard algorithm for $m$-way trees
Search for L
**Insertion in 2-3-trees**

- We can start by following the generic algorithm for $m$-way trees
- Search for the value $l$ we want to insert
- If found, stop (no duplicates)
- If not found, insert at the **leaf** we reached
Example: insert B
Example: insert B
Example: result

Diagram of a hierarchical structure with nodes labeled A to P.
Insertion in 2-3-trees

• But what if there is no space at the leaf (overflow)?

• The standard algorithm will insert a child at the leaf
  - But this violates the depth property!
  - The new leaf is not at the same level

• Different strategy
  - **split** the overflowed node into two nodes
  - pass the middle value to the parent (separator of the two nodes)

• The middle value might overflow the parent
  - Same procedure: split and send the middle value up
Example: insert M
Example: insert M

M overflows this node.
The node is split in two and Lis passed to the parent node
Example: insert M

Loverflows this node

A B
D
E F
I
K
M
O P

H
J N
L
Example: insert M

The node is split in two and Lis passed up to the parent
Example: result

Lis inserted in the root node
Example: insert Q

Q overflows this node
Example: insert Q

This node is split up and P is passed up
Example: result
Example: insert R

R is inserted in the node with Q where there is space.
Insertion in 2-3-trees

- The root might also overflow
- Same procedure
  - Split it
  - The middle value moves up, creating a new root
- This is the only operation that increases the tree's height
  - It increases the depth of all nodes simultaneously
  - 2-3-trees grow at the root, not at the leaves!
Example: insert S

S overflows this node
Example: insert S

This node is split and R is sent up
Example: insert S
Example: insert S

This node is split up and P is sent up
Example: insert S

P overflows the root

A B
D
E F
I
J
K
M
N
O
P
Q
R
S
Example: result

The root splits and L becomes the new root
Complexity of insertion

- We traverse the tree
  - From the root to a leaf when searching
  - From the leaf back to the root while splitting
- Each split takes constant time
  - We do at most $h + 1$ of them
- So in total $O(h) = O(\log n)$ steps
  - Recall, the tree is balanced
2-4 trees

• A 2-4 tree (or 2-3-4 tree) is a 4-way search tree with 2 extra properties

• Size property
  - Each node contains between 1 and 3 values
    (so each internal node contains between 2 and 4 children)

• Depth property
  - All leaves have the same depth (lie on the same level)

• Such trees are balanced
  - \( h = O(\log n) \)
  - Proof: exercise
Insertion in 2-4 trees

- Same as for 2-3-trees
  - Search for the value
  - Insert at a leaf

- In case of an overflow (5-node)
  - Split it into a 3-node and a 2-node
  - Move the separator value $k_3$ to the parent
Overflow at a 5-node
The separating value is sent to the parent node

\[ v = u_2 \]

\[ k_3 \]

\[ v_1 \quad v_2 \quad v_3 \quad v_4 \quad v_5 \]
Node replaced with a 3-node and a 2-node
Example: insert 4
Example: insert 6
Example: insert 12

4 6 12
Example: insert 15 - overflow
Creation of new root node
Split
Example: insert 3
Example: insert 5 - overflow
5 is sent to the parent node

```
3 4 6
```

```
12
```

```
5
```

```
15
```
Split

Diagram:

- Node 5 connected to node 3, node 4, and node 12.
- Node 12 connected to node 6 and node 15.
Example: insert 10
Example: insert 8
Example

Inserted 11, 13 and 14.
Example: insert 17 - overflow
Split and send 15 to the parent node
The root overflows
Creation of new root
Split

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Split

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Split

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Split

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Split

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Split

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Split

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Final tree
Complexity

• Same as for 2-3-trees
  - At most $h$ splits
  - Each split is constant time

• $O(\log n)$
  - Because the tree is balanced
Removal in 2-4 trees

- To remove a value $k_i$ from an **internal** node
  - Replace with its **predecessor** (or its **successor**)
  - Right-most value in the $i$-th subtree
  - Similar to the BST case of nodes with two children

- To remove a value from a **leaf**
  - We simply remove it
  - But it might violate the **size** property (**underflow**)
Fixing underflows

Two strategies for fixing an underlow at \( \nu \)

- Is there an **immediate sibling** \( \omega \) with a “spare” value? (2 or 3 values)
- If so, we do a **transfer** operation
  - Move a value of \( \omega \) to its parent \( \omega \)
  - Move a value of the parent \( \omega \) to \( \nu \)
- If not, we do a **fusion** operation
  - Merge \( \nu \) and \( \omega \), creating a new node \( \nu' \)
  - Move a value from the parent \( \omega \) to \( \nu' \)
  - This might **underflow the parent**, continue the same procedure there
Initial tree

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Initial tree
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Initial tree
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Initial tree
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Remove 4
Transfer
After the transfer
Remove 12
Remove 12
Fusion of and
After the fusion

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5
6
8 10
11
13 14
15
17
```
Remove 13
After the removal of 13

Diagram:

- Node 11
  - Node 6
    - Node 5
    - Node 8
    - Node 10
  - Node 15
    - Node 14
    - Node 17
Remove 14 - underflow
Fusion
Underflow at

```
  11
  / \   
 6   u
  |
5  8 10
  |
15 17
```
Fusion

5 6 8 10 11 u 15 17
Remove the root
Final tree
Readings

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