Weighted graphs

Κ08 Δομές Δεδομένων και Τεχνικές Προγραμματισμού
Κώστας Χατζηκοκολάκης
Weighted graphs

• Graphs with numbers, called weights, attached to each edge
  - Often restricted to non-negative

• Directed or undirected

• Examples
  - Distance between cities
  - Cost of flight between airports
  - Time to send a message between routers
Weighted graphs

• Adjacency matrix representation

\[ T[i, j] = \begin{cases} w_{i,j} & \text{if } i, j \text{ are connected} \\ \infty & \text{if } i \neq j \text{ are not connected} \\ 0 & \text{if } i = j \end{cases} \]

• Similarly for adjacency lists
Example weighted graph
# Example weighted graph

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
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<tbody>
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</table>

Adjacency matrix
Shortest paths

• The **length** of a path is the **sum of the weights** of its edges

• Very common problem
  - find the **shortest path** from \( s \) to \( d \)

• Examples
  - Shortest route between cities
  - Cheapest connecting flight
  - Fastest network route
  - ...
Shortest path from vertex 1 to vertex 5

1 - 6 - 5

Distance: 7
Shortest path problem

Two main variants:

• **Single source** $s$
  - Find the shortest path from $s$ to each node
  - **Dijkstra's** algorithm
    - Only for non-negative weights (important!)

• **All-pairs**
  - Find the shortest path between all pairs $s, d$
  - **Floyd-Warshall** algorithm
    - Any weights
Dijkstra's algorithm

Main ideas:

• Keep a set \( W \) of \textbf{visited} nodes
  - Start with \( W = \{s\} \) (or \( W = \{\} \))

• Keep a matrix \( \Delta[u] \)
  - Minimum distance from \( s \) to \( u \) \textbf{passing only through} \( W \)
  - Start with \( \Delta[u] = T[s, u] \) (or \( \Delta[s] = 0, \Delta[u] = \infty \))

• At each step we \textbf{enlarge} \( W \) by adding a \textbf{new vertex} \( w \notin W \)
  - \( w \) is the one with \textbf{minimum} \( \Delta[w] \)
Dijkstra's algorithm

Main ideas:

• Adding $w$ to $W$ might affect $\Delta[u]$
  - For some neighbour $u$ of $w$

• We might now have a shorter path to $u$ passing through $w$
  - Of the form $s \rightarrow \ldots \rightarrow w \rightarrow u$
  - If $\Delta[u] > \Delta[w] + T[w, u]$

• In this case we update $\Delta$
  - $\Delta[u] = \Delta[w] + T[w, u]$
Example graph
Expanding the vertex set $w$ in stages

<table>
<thead>
<tr>
<th>Stage</th>
<th>$W$</th>
<th>$V-W$</th>
<th>$w$</th>
<th>$\Delta(w)$</th>
<th>$\Delta(1)$</th>
<th>$\Delta(2)$</th>
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</tr>
</thead>
<tbody>
<tr>
<td>Start</td>
<td>${1}$</td>
<td>${2,3,4,5,6}$</td>
<td>-</td>
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<td>0</td>
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</table>

Diagram: A graph with vertices labeled 1 to 6 and edges labeled with respective weights.
Expanding the vertex set \( w \) in stages

\( W=2 \) is chosen for the second stage.

<table>
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<tr>
<th>Stage</th>
<th>( W )</th>
<th>( V-W )</th>
<th>( w )</th>
<th>( \Delta(w) )</th>
<th>( \Delta(1) )</th>
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</table>

![Graph](image_url)
Expanding the vertex set $w$ in stages

<table>
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<tr>
<th>Stage</th>
<th>W</th>
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<th>w</th>
<th>Δ(w)</th>
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![Graph diagram with nodes and edges illustrating the expansion process.](image)
Expanding the vertex set $w$ in stages

$W=6$ is chosen for the third stage.

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Expanding the vertex set w in stages

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The diagram shows the expansion of the vertex set w in stages, with edges indicating connections and weights.
Expanding the vertex set $w$ in stages

$W=4$ is chosen for the fourth stage.

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Expanding the vertex set \( w \) in stages

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</table>

![Diagram of the graph with vertex set expansions](image-url)
Expanding the vertex set w in stages

W=3 is chosen for the fifth stage.

<table>
<thead>
<tr>
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<th>W</th>
<th>V-W</th>
<th>w</th>
<th>Δ(w)</th>
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</table>
Expanding the vertex set $w$ in stages

$W=5$ is chosen for the sixth stage.

<table>
<thead>
<tr>
<th>Stage</th>
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<th>$\Delta(w)$</th>
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![Graph diagram](image-url)
// Δεδομένα
src : αρχικός κόμβος
dest : τελικός κόμβος

// Πληροφορίες που κρατάμε για κάθε κόμβο v
W[u] : 1 αν ο u είναι στο σύνολο W, Θ διαφορετικά
dist[u] : ο πίνακας Δ
prev[u] : ο προηγούμενος του v στο βέλτιστο μονοπάτι

// Αρχικοποίηση: W={} (εναλλακτικά μπορούμε και W={src})
for each vertex u in Graph
  dist[u] = INT_MAX // infinity
  prev[u] = NULL
  W[u] = 0

dist[src] = 0
Dijkstra's algorithm in pseudocode

// Κύριως αλγόριθμος
while true
    w = vertex with minimum dist[w], among those with W[w] = 0

    W[w] = 1
    if w == dest
        stop
        // optimal cost = dist[dest]
        // optimal path = dest <- prev[dest] <- ... <- src (inverse)

    for each neighbor u of w
        if W[u] == 1
            continue
        alt = dist[w] + weight(w,u)  // κόστος του src -> ... -> w
        if alt < dist[u]  // καλύτερο από πριν, update
            dist[u] = alt
            prev[u] = w
Using a priority queue

- Finding the $w \not\in W$ with minimum $\Delta[w]$ is slow
  - $O(n)$ time

- But we can use a priority queue for this!
  - We only keep vertices $w \not\in W$ in the queue
  - They are compared based on their $\Delta[w]$
    (each $w$ has “priority” $\Delta[w]$)

- Careful when $\Delta[w]$ is modified!
  - Either use a priority queue that allows updates
  - Or insert multiple copies of each $w$ with different priorities
    - the queue might contain already visited vertices: ignore them
Dijkstra's algorithm with priority queue

// Δεδομένα
src : αρχικός κόμβος
dest : τελικός κόμβος

// Πληροφορίες που κρατάμε για κάθε κόμβο u
W[u] : 1 αν ο v είναι στο σύνολο W, θ διαφορετικά
dist[u] : ο πίνακας Δ
t[prev] : ο προηγούμενος του v στο βέλτιστο μονοπάτι
pq : Priority queue, εισάγουμε tuples {u,dist[u]}

// Αρχικοποίηση: W={} (εναλλακτικά μπορούμε και W={src})
prev[src] = NULL
dist[src] = 0
pqueue_insert(pq, {src, 0})  // dist[src] = 0
Dijkstra's algorithm with priority queue

```plaintext
// Κύριως αλγόριθμος
while pq is not empty
    w = pqueue_max(pq)  // w with minimal dist[u]
    pqueue_remove_max(pq)

    if exists(W[w])  // το w μπορεί να υπάρχει πολλές φορές στην ο
        continue  // δεν κάνουμε replace), και να είναι ήδη vis
    W[w] = 1
    if w == dest
        stop  // optimal cost/path same as before

    for each neighbor u of w
        if exists(W[u])
            continue
        alt = dist[w] + weight(w,u)  // cost of src->...->w->u
        if !exists(dist[u]) OR alt < dist[u]
            dist[u] = alt
            prev[u] = w
            pqueue_insert(pq, {u,alt})  // προαιρετικά: replace αν υπ

    stop // pq άδειασε πριν βρούμε το dest => δεν υπάρχει μονοπάτι
```
Notation

• $s \rightarrow d$
  - Direct step step from $s$ to $d$

• $s \underset{W}{\rightarrow} d$
  - Multiple steps $s \rightarrow \ldots \rightarrow d$
  - All intermediate steps belong to the set $W \subseteq V$

• $s \underset{W}{\rightarrow} d$
  - Shortest path among all $s \underset{W}{\rightarrow} d$
  - So $s \underset{V}{\rightarrow} d$ is the overall shortest one
Proof of correctness

• We need to prove that $\Delta[u]$ is the minimum distance to $u$
  - after the algorithm finishes

• But it's much easier to reason step by step
  - we need a property that holds at every step
  - this is called an invariant (property that never changes)
Proof of correctness

Invariant of Dijkstra's algorithm

- $\Delta[u]$ is the cost of the shortest path passing only through $W$
- And the shortest overall when $u \in W$

Formally:

1. For all $u \in V$ the path $s \xrightarrow{W} u$ has cost $\Delta[u]$
2. For all $u \in W$ the path $s \xrightarrow{V} u$ has cost $\Delta[u]$

Proof: induction on the size of $W$, for both (1), (2) together
Proof of correctness

Base case $W = \{s\}$

- Trivial, the only path $s \xrightarrow{W} u$ is the direct one $s \rightarrow u$
- For (1): its cost is exactly $T[s, u] = \Delta[u]$
  - initial value of $\Delta[u]$
- For (2): the only $u \in W$ is $s$ itself
Proof of correctness

Inductive case

- Assume $|\mathcal{W}| = k$ and (1),(2) hold

- The algorithm
  - Updates $\mathcal{W}$, adding a new vertex $w \notin \mathcal{W}$
  - Updates $\Delta[u]$ for all neighbours $u$ of $w$

- Let $\mathcal{W}', \Delta'$ be the values after the update

- Show that (1),(2) still hold for $\mathcal{W}', \Delta'$
Proof of correctness

We start showing that (2) still holds for $W'$, $\Delta'$

- The interesting vertex is the $w$ we just added
  - Vertices $u \neq w$ are trivial from the induction hypothesis

- Show: $s \xrightarrow{V} w$ has cost $\Delta'[w]$
  - Note: $\Delta'[w] = \Delta[w]$ (we do not update $\Delta[w]$)
  - We already know that $s \xrightarrow{W} w$ has cost $\Delta[w]$ (ind. hyp)
  - So we just need to prove that there is no better path outside $W$
Proof of correctness

- Assuming such path exists, let \( r \) be its **first** vertex outside \( W \)
  - So the path \( s \xrightarrow{W} r \xrightarrow{V} w \) has cost \( c < \Delta[w] \)
  - So the path \( s \xrightarrow{W} r \) has cost at most \( c < \Delta[w] \) (no negative weights!)
  - So \( \Delta[r] < \Delta[w] \)

- **Impossible!** We chose \( w \) to be the one with min \( \Delta[w] \)
Proof of correctness

It remains to show (1) for $W'$, $\Delta'$

• Take some arbitrary $u$
  
  - Let $c$ be the cost of $s \xrightarrow{W'} u$
  
  - Show: $c = \Delta'[u]$

• Three cases for the optimal path $s \xrightarrow{W'} u$

• Case 1: the path does not pass through $w$
  
  - So it is of the form $s \xrightarrow{W} u$
  
  - This path has cost $\Delta[u]$ (induction hypothesis)
  
  - No update: $\Delta'[u] = \Delta[u] = c$
Proof of correctness

• Case 2: \( w \) is right before \( u \)
  - So the path is of the form \( s \xrightarrow{W} w \rightarrow u \)
  - The cost of \( s \xrightarrow{W} w \) is \( \Delta[w] \) (induction hypothesis)
  - So \( c = \Delta[w] + T[w, u] \)
  - So the algorithm will set \( \Delta'[u] = \Delta[w] + T[w, u] \)
    when updating the neighbours of \( w \)
  - So \( c = \Delta'[u] \)
Proof of correctness

- Case 3: some other $x$ appears after $w$ in the path
  - So the path is of the form $s \xrightarrow{W} w \rightarrow x \xrightarrow{W} u$
  - But the path $s \xrightarrow{W} w \rightarrow x$ is no shorter than $s \xrightarrow{W} x$
    - From the induction hypothesis for $x \in W$
  - So $s \xrightarrow{W} x \rightarrow u$ is also optimal, reducing to case 1!
Complexity

Without a priority queue:

- Initialization stage: loop over vertices: \( O(n) \)
- The while-loop adds one vertex every time: \( n \) iterations
- Finding the new vertex loops over vertices: \( O(n) \)
  - same for updating the neighbours
- So total \( O(n^2) \) time
Complexity

With a priority queue:

- Initialization stage: loop over vertices, so $O(n)$
- Count the number of updates (steps in the inner loop)
  - Once for every neighbour of every node: $e$ total
  - Each update is $O(\log n)$ (at most $n$ elements in the queue)
- Total $O(e \log n)$
  - Assuming a connected graph ($e \geq n$)
  - And an implementation using adjacency lists
- Only good for sparse graphs!
  - But $O(n \log n)$ can be hugely better than $O(n^2)$
The all-pairs shortest path problem

• Find the shortest path between all pairs \( s, d \)

• **Floyd-Warshall** algorithm

• Any weights
  - Even negative
  - But no **negative loops** (why?)
The all-pairs shortest path problem

Main idea

• At each step we compute the shortest path through a subset of vertices
  - Similarly to $W$ in Dijkstra
  - But now the set at step $k$ is $W_k = \{1, \ldots, k\}$
    ◦ Vectors are numbered in any order

• Step $k$: the cost of $i \xrightarrow{W_k} j$ is $A_k[i, j]$
  - Similar to $\Delta$ in Dijkstra (but for all pairs $i, j$ of nodes)
Floyd-Warshall algorithm

- The algorithm at each step computes $A_k$ from $A_{k-1}$

- Initial step $k = 0$
  - Start with $A_0[i, j] = T[i, j]$
  - Only direct paths are allowed
Floyd-Warshall algorithm

$k$-th iteration: the optimal $i \xrightarrow{W_k} j$ either **passes thorough** $k$ or not.

$$A_k[i, j] = \min \begin{cases} A_{k-1}[i, j] \\ A_{k-1}[i, k] + A_{k-1}[k, j] \end{cases}$$
Floyd-Warshall algorithm in pseudocode

```c
void floyd_warshall() {
    for (int i = 0; i <= size-1; i++)
        for (int j = 0; j <= size-1; j++)
            A[i][j] = weight(i, j)

    for (int i = 0; i <= size-1; i++)
        A[i][i] = 0;

    for (int k = 0; k <= size-1; k++)
        // Compute A_k from A_{k-1}
        for (int i = 0; i <= size-1; i++)
            for (int j = 0; j <= size-1; j++)
}
```

A is the current $A_k$ at every step $k$. 
Complexity

• Three simple loops of $n$ steps
• So $O(n^3)$
• **Not** better than $n$ executions of Dijkstra in complexity
  - But much simpler
  - Often faster in practice
  - And works for **negative** weights
Readings

• T. A. Standish. *Data Structures, Algorithms and Software Principles in C*. Chapter 10