Weighted graphs

Κ08 Δομές Δεδομένων και Τεχνικές Προγραμματισμού
Κώστας Χατζηκοκολάκης
Weighted graphs

- Graphs with numbers, called **weights**, attached to each edge
  - Often restricted to **non-negative**

- Directed or undirected

- Examples
  - **Distance** between cities
  - **Cost** of flight between airports
  - **Time** to send a message between routers
Weighted graphs

- Adjacency matrix representation

\[ T[i, j] = \begin{cases} 
  w_{i,j} & \text{if } i, j \text{ are connected} \\
  \infty & \text{if } i \neq j \text{ are not connected} \\
  0 & \text{if } i = j 
\end{cases} \]

- Similarly for adjacency lists
Example weighted graph
Example weighted graph

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
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<tbody>
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</table>

Adjacency matrix
Shortest paths

• The **length** of a path is the **sum of the weights** of its edges

• Very common problem
  - find the **shortest path** from \( s \) to \( d \)

• Examples
  - Shortest route between cities
  - Cheapest connecting flight
  - Fastest network route
  - ...
Shortest path from vertex 1 to vertex 5
Shortest path problem

Two main variants:

- **Single source s**
  - Find the shortest path from \( s \) to each node
  - **Dijkstra's** algorithm
    - Only for **non-negative** weights (important!)

- **All-pairs**
  - Find the shortest path between all pairs \( s, d \)
  - **Floyd-Warshall** algorithm
    - Any weights
Dijkstra's algorithm

Main ideas:

- Keep a set $W$ of *visited* nodes
  - Start with $W = \{s\}$ (or $W = \{}\)

- Keep a matrix $\Delta[u]$
  - Minimum distance from $s$ to $u$ *passing only through* $W$
  - Start with $\Delta[u] = T[s, u]$ (or $\Delta[s] = 0, \Delta[u] = \infty$)

- At each step we *enlarge* $W$ by adding a *new vertex* $w \notin W$
  - $w$ is the one with minimum $\Delta[w]$
Dijkstra's algorithm

Main ideas:

• Adding $w$ to $W$ might affect $\Delta[u]$
  - For some neighbour $u$ of $w$

• We might now have a shorter path to $u$ passing through $w$
  - Of the form $s \rightarrow \ldots \rightarrow w \rightarrow u$
  - If $\Delta[u] > \Delta[w] + T[w, u]$

• In this case we update $\Delta$
  - $\Delta[u] = \Delta[w] + T[w, u]$
Example graph
Expanding the vertex set \( w \) in stages

<table>
<thead>
<tr>
<th>Stage</th>
<th>W</th>
<th>V-W</th>
<th>w</th>
<th>( \Delta(w) )</th>
<th>( \Delta(1) )</th>
<th>( \Delta(2) )</th>
<th>( \Delta(3) )</th>
<th>( \Delta(4) )</th>
<th>( \Delta(5) )</th>
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<tbody>
<tr>
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</tbody>
</table>

Graph showing the expansion of vertex set \( w \) in stages.
Expanding the vertex set w in stages

W=2 is chosen for the second stage.

<table>
<thead>
<tr>
<th>Stage</th>
<th>W</th>
<th>V-W</th>
<th>w</th>
<th>Δ(w)</th>
<th>Δ(1)</th>
<th>Δ(2)</th>
<th>Δ(3)</th>
<th>Δ(4)</th>
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</table>
Expanding the vertex set $w$ in stages

<table>
<thead>
<tr>
<th>Stage</th>
<th>W</th>
<th>V-W</th>
<th>$w$</th>
<th>$\Delta(w)$</th>
<th>$\Delta(1)$</th>
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![Diagram showing the expanding process]
Expanding the vertex set \( w \) in stages

\( W=6 \) is chosen for the third stage.

<table>
<thead>
<tr>
<th>Stage</th>
<th>( W )</th>
<th>V-W</th>
<th>( w )</th>
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<th>( \Delta(1) )</th>
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Expanding the vertex set $w$ in stages

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</table>
Expanding the vertex set $w$ in stages

$W=4$ is chosen for the fourth stage.

<table>
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<td>13</td>
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</table>

![Graph diagram showing the vertex set expansion stages](attachment:image.png)
Expanding the vertex set $w$ in stages

$W=3$ is chosen for the fifth stage.

<table>
<thead>
<tr>
<th>Stage</th>
<th>$W$</th>
<th>$V-W$</th>
<th>$w$</th>
<th>$\Delta(w)$</th>
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</table>

![Graph](image.png)
Expanding the vertex set $w$ in stages

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</table>
Expanding the vertex set $w$ in stages

$W=5$ is chosen for the sixth stage.

<table>
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<tr>
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<th>V-W</th>
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<th>$\Delta(w)$</th>
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</table>

![Graph diagram](image)
Expanding the vertex set $w$ in stages

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</tr>
</tbody>
</table>

![Graph Illustration](image)
Dijkstra's algorithm in pseudocode

// Δεδομένα
src : αρχικός κόμβος
dest : τελικός κόμβος

// Πληροφορίες που κρατάμε για κάθε κόμβο v
W[u] : 1 αν ο u είναι στο σύνολο W, θ διαφορετικά
dist[u] : ο πίνακας δ
prev[u] : ο προηγούμενος του v στο βέλτιστο μονοπάτι

// Αρχικοποίηση: W={} (εναλλακτικά μπορούμε και W={src})
for each vertex u in Graph
  dist[u] = INT_MAX // infinity
  prev[u] = NULL
  W[u] = 0

dist[src] = 0
Dijkstra's algorithm in pseudocode

// Κυρίως αλγόριθμος
while true
  w = vertex with minimum dist[w], among those with W[w] = 0

  W[w] = 1
  if w == dest
    stop
    // optimal cost = dist[dest]
    // optimal path = dest <- prev[dest] <- ... <- src (inverse)

  for each neighbor u of w
    if W[u] == 1
      continue
    alt = dist[w] + weight(w,u)  // κόστος του src -> ... -> w
    if alt < dist[u]
      dist[u] = alt
      prev[u] = w
Using a priority queue

- Finding the \( w \notin W \) with minimum \( \Delta[w] \) is slow
  - \( O(n) \) time

- But we can use a **priority queue** for this!
  - We only keep vertices \( w \notin W \) in the queue
  - They are compared based on their \( \Delta[w] \)
    (each \( w \) has “priority” \( \Delta[w] \))

- Careful when \( \Delta[w] \) is modified!
  - Either use a priority queue that allows **updates**
  - Or insert multiple copies of each \( w \) with **different priorities**
    - the queue might contain **already visited** vertices: ignore them
Dijkstra's algorithm with priority queue

// Δεδομένα
src : αρχικός κόμβος
dest : τελικός κόμβος

// Πληροφορίες που κρατάμε για κάθε κόμβο u
W[u] : 1 αν o v είναι στο σύνολο W, θ διαφορετικά
dist[u] : o πίνακας Δ
prev[u] : o προηγούμενος του v στο βέλτιστο μονοπάτι
pq : Priority queue, εισάγουμε tuples {u,dist[u]}
    συγκρίνοντας με βάση το dist[u]

// Αρχικοποίηση: W={} (εναλλακτικά μπορούμε και W={src})
prev[src] = NULL
dist[src] = 0
pq_insert(pq, {src,0}) // dist[src] = 0
Dijkstra's algorithm with priority queue

```c
// Κυρίως αλγόριθμος
while pq is not empty
    w = pqueue_max(pq)  // w with minimal dist[u]
    pqueue_remove_max(pq)

    if exists(W[w])     // το w μπορεί να υπάρχει πολλές φορές στην ο 
        continue     // δεν κάνουμε replace), και να είναι ήδη vis
    W[w] = 1
    if w == dest
        stop        // optimal cost/path same as before

    for each neighbor u of w
        if exists(W[u])
            continue
        alt = dist[w] + weight(w,u)  // cost of src->...->w->u
        if !exists(dist[u]) OR alt < dist[u]
            dist[u] = alt
            prev[u] = w
            pqueue_insert(pq, {u,alt})  // προαιρετικά: replace αν υπ
    stop // pq άδειασε πριν βρούμε το dest => δεν υπάρχει μονοπάτι
```
Notation

• $s \rightarrow d$
  - Direct step step from $s$ to $d$

• $s \xrightarrow{W} d$
  - Multiple steps $s \rightarrow \ldots \rightarrow d$
  - All intermediate steps belong to the set $W \subseteq V$

• $s \xrightarrow{W} d$
  - Shortest path among all $s \xrightarrow{W} d$
  - So $s \xrightarrow{V} d$ is the overall shortest one
Proof of correctness

• We need to prove that $\Delta[u]$ is the minimum distance to $u$
  - after the algorithm finishes

• But it's much easier to reason step by step
  - we need a property that holds at every step
  - this is called an invariant (property that never changes)
Proof of correctness

Invariant of Dijkstra's algorithm

• $\Delta[u]$ is the cost of the shortest path passing only through $W$

• And the shortest overall when $u \in W$

Formally:

1. For all $u \in V$ the path $s \xrightarrow{W} u$ has cost $\Delta[u]$

2. For all $u \in W$ the path $s \xrightarrow{V} u$ has cost $\Delta[u]$

Proof: induction on the size of $W$, for both (1), (2) together
Proof of correctness

Base case $W = \{s\}$

- Trivial, the only path $s \xrightarrow{W} u$ is the direct one $s \rightarrow u$
- For (1): its cost is exactly $T[s, u] = \Delta[u]$
  - initial value of $\Delta[u]$
- For (2): the only $u \in W$ is $s$ itself
Proof of correctness

Inductive case

• Assume $|W| = k$ and (1),(2) hold

• The algorithm
  - Updates $W$, adding a new vertex $w \not\in W$
  - Updates $\Delta[u]$ for all neighbours $u$ of $w$

• Let $W'$, $\Delta'$ be the values after the update

• Show that (1),(2) still hold for $W'$, $\Delta'$
Proof of correctness

We start showing that (2) still holds for $W', \Delta'$

• The interesting vertex is the $w$ we just added
  - Vertices $u \neq w$ are trivial from the induction hypothesis

• Show: $s \xrightarrow{V} w$ has cost $\Delta'[w]
  - Note: $\Delta'[w] = \Delta[w]$ (we do not update $\Delta[w]$)
  - We already know that $s \xrightarrow{W} w$ has cost $\Delta[w]$ (ind. hyp)
  - So we just need to prove that there is no better path outside $W$
Proof of correctness

• Assuming such path exists, let \( r \) be its \textbf{first} vertex outside \( W \)
  
  \[ \begin{align*}
  \text{- So the path } s \xrightarrow{W} r \xrightarrow{V} w \text{ has cost } c < \Delta[w] \\
  \text{- So the path } s \xrightarrow{W} r \text{ has cost at most } c < \Delta[w] \text{ (no negative weights!)} \\
  \text{- So } \Delta[r] < \Delta[w]
  \end{align*} \]

• \textbf{Impossible!} We chose \( w \) to be the one with min \( \Delta[w] \)
Proof of correctness

It remains to show (1) for $W'$, $\Delta'$

- Take some arbitrary $u$
  - Let $c$ be the cost of $s \xrightarrow{W'} u$
  - Show: $c = \Delta'[u]$

- Three cases for the optimal path $s \xrightarrow{W'} u$

- Case 1: the path does not pass through $w$
  - So it is of the form $s \xrightarrow{W} u$
  - This path has cost $\Delta[u]$ (induction hypothesis)
  - No update: $\Delta'[u] = \Delta[u] = c$
Proof of correctness

• Case 2: \( w \) is right before \( u \)
  - So the path is of the form \( s \xrightarrow{W} w \rightarrow u \)
  - The cost of \( s \xrightarrow{W} w \) is \( \Delta[w] \) (induction hypothesis)
  - So \( c = \Delta[w] + T[w,u] \)
  - So the algorithm will set \( \Delta'[u] = \Delta[w] + T[w,u] \) when updating the neighbours of \( w \)
  - So \( c = \Delta'[u] \)
Proof of correctness

• Case 3: some other $x$ appears after $w$ in the path
  - So the path is of the form $s \xrightarrow{W} w \rightarrow x \xrightarrow{W} u$
  - But the path $s \xrightarrow{W} w \rightarrow x$ is no shorter than $s \xrightarrow{W} x$
    - From the induction hypothesis for $x \in W$
  - So $s \xrightarrow{W} x \rightarrow u$ is also optimal, reducing to case 1!
Complexity

Without a priority queue:

• Initialization stage: loop over vertices: $O(n)$

• The while-loop adds one vertex every time: $n$ iterations

• Finding the new vertex loops over vertices: $O(n)$
  - same for updating the neighbours

• So total $O(n^2)$ time
Complexity

With a priority queue:

- Initialization stage: loop over vertices, so $O(n)$

- Count the number of updates (steps in the inner loop)
  - Once for every neighbour of every node: $e$ total
  - Each update is $O(\log n)$ (at most $n$ elements in the queue)

- Total $O(e \log n)$
  - Assuming a connected graph ($e \geq n$)
  - And an implementation using adjacency lists

- Only good for sparse graphs!
  - But $O(n \log n)$ can be hugely better than $O(n^2)$
The all-pairs shortest path problem

- Find the shortest path between all pairs $s, d$
- **Floyd-Warshall** algorithm
- Any weights
  - Even negative
  - But no **negative loops** (why?)
The all-pairs shortest path problem

Main idea

• At each step we compute the shortest path through a subset of vertices
  - Similarly to $W$ in Dijkstra
  - But now the set at step $k$ is $W_k = \{1, \ldots, k\}$
    ◦ Vertices are numbered in any order

• Step $k$: the cost of $i \xrightarrow{W_k} j$ is $A_k[i, j]$
  - Similar to $\Delta$ in Dijkstra (but for all pairs $i, j$ of nodes)
Floyd-Warshall algorithm

• The algorithm at each step computes $A_k$ from $A_{k-1}$

• Initial step $k = 0$
  - Start with $A_0[i, j] = T[i, j]$
  - Only direct paths are allowed
Floyd-Warshall algorithm

$k$-th iteration: the optimal $i \xrightarrow{W_k} j$ either passes thorough $k$ or not.

$$A_k[i, j] = \min \left\{ A_{k-1}[i, j], A_{k-1}[i, k] + A_{k-1}[k, j] \right\}$$
Floyd-Warshall algorithm in pseudocode

```c
void floyd_warshall() {
    for (int i = 0; i <= size-1; i++)
        for (int j = 0; j <= size-1; j++)
            A[i][j] = weight(i, j)

    for (int i = 0; i <= size-1; i++)
        A[i][i] = 0;

    for (int k = 0; k <= size-1; k++)
        // Compute A_k from A_{k-1}
        for (int i = 0; i <= size-1; i++)
            for (int j = 0; j <= size-1; j++)
}
```

A is the current $A_k$ at every step $k$. 
Complexity

• Three simple loops of $n$ steps
• So $O(n^3)$

• **Not** better than $n$ executions of Dijkstra in **complexity**
  - But much simpler
  - Often faster in practice
  - And works for **negative** weights
Readings

• T. A. Standish. *Data Structures, Algorithms and Software Principles in C.* Chapter 10

• A. V. Aho, J. E. Hopcroft and J. D. Ullman. *Data Structures and Algorithms.* Chapters 6 and 7