Weighted graphs

Κ08 Δομές Δεδομένων και Τεχνικές Προγραμματισμού
Κώστας Χατζηκοκολάκης
Weighted graphs

• Graphs with numbers, called **weights**, attached to each edge
  - Often restricted to **non-negative**

• Directed or undirected

• Examples
  - **Distance** between cities
  - **Cost** of flight between airports
  - **Time** to send a message between routers
Weighted graphs

- Adjacency matrix representation

$$T[i, j] = \begin{cases} 
    w_{i,j} & \text{if } i, j \text{ are connected} \\
    \infty & \text{if } i \neq j \text{ are not connected} \\
    0 & \text{if } i = j 
\end{cases}$$

- Similarly for adjacency lists
Example weighted graph
**Example weighted graph**

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
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</tbody>
</table>

**Adjacency matrix**
Shortest paths

• The **length** of a path is the **sum of the weights** of its edges

• Very common problem
  - find the **shortest path** from \( s \) to \( d \)

• Examples
  - Shortest route between cities
  - Cheapest connecting flight
  - Fastest network route
  - ...
Shortest path from vertex 1 to vertex 5
Shortest path problem

Two main variants:

• **Single source** \( s \)
  - Find the shortest path from \( s \) to each node
  - **Dijkstra's** algorithm
    ◦ Only for **non-negative** weights (important!)

• **All-pairs**
  - Find the shortest path between all pairs \( s, d \)
  - **Floyd-Warshall** algorithm
    ◦ Any weights
Dijkstra's algorithm

Main ideas:

• Keep a set \( W \) of **visited** nodes
  - Start with \( W = \{s\} \)  \((\text{or } W = \{\})\)

• Keep a matrix \( \Delta[u] \)
  - Minimum distance from \( s \) to \( u \) **passing only through** \( W \)
  - Start with \( \Delta[u] = T[s, u] \)  \((\text{or } \Delta[s] = 0, \Delta[u] = \infty)\)

• At each step we **enlarge** \( W \) by adding a **new vertex** \( w \notin W \)
  - \( w \) is the one with **minimum** \( \Delta[w] \)
Dijkstra's algorithm

Main ideas:

• Adding $w$ to $W$ might affect $\Delta[u]$
  - For some neighbour $u$ of $w$

• We might now have a shorter path to $u$ passing through $w$
  - Of the form $s \rightarrow \ldots \rightarrow w \rightarrow u$
    - If $\Delta[u] > \Delta[w] + T[w, u]$

• In this case we update $\Delta$
  - $\Delta[u] = \Delta[w] + T[w, u]$
Example graph
Expanding the vertex set \( w \) in stages

<table>
<thead>
<tr>
<th>Stage</th>
<th>( W )</th>
<th>( V-W )</th>
<th>( w )</th>
<th>( \Delta(w) )</th>
<th>( \Delta(1) )</th>
<th>( \Delta(2) )</th>
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<tr>
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![Graph](image_url)
Expanding the vertex set $w$ in stages

$W=2$ is chosen for the second stage.

<table>
<thead>
<tr>
<th>Stage</th>
<th>W</th>
<th>V-W</th>
<th>w</th>
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![Graph](image.png)
Expanding the vertex set $w$ in stages

$W=6$ is chosen for the third stage.

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Diagram:

- Vertices and edges are labeled with numbers and weights,
- Red edges indicate the selected vertex at each stage.
- The stage is indicated by the color and number of the vertex.

Note: The table and diagram illustrate the process of expanding the vertex set $w$ in stages, with $W=6$ chosen for the third stage.
Expanding the vertex set $w$ in stages

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Expanding the vertex set $w$ in stages

$W=4$ is chosen for the fourth stage.

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Expanding the vertex set \( w \) in stages

\( W=3 \) is chosen for the fifth stage.

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Expanding the vertex set $w$ in stages

$W=5$ is chosen for the sixth stage.

<table>
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![Graph diagram](image-url)
Dijkstra's algorithm in pseudocode

// Δεδομένα
src : αρχικός κόμβος
dest : τελικός κόμβος

// Πληροφορίες που κρατάμε για κάθε κόμβο v
W[u] : 1 αν o u είναι στο σύνολο W, θ διαφορετικά
dist[u] : o πίνακας Δ
prev[u] : o προηγούμενος του v στο βέλτιστο μονοπάτι

// Αρχικοποίηση: W={} (εναλλακτικά μπορούμε και W={src})
for each vertex u in Graph
    dist[u] = INT_MAX // infinity
    prev[u] = NULL
    W[u] = θ

dist[src] = 0
Dijkstra's algorithm in pseudocode

```plaintext
// Κυρίως αλγόριθμος
while true
    w = vertex with minimum dist[w], among those with W[w] = 0
    W[w] = 1
    if w == dest
        stop
        // optimal cost = dist[dest]
        // optimal path = dest <- prev[dest] <- ... <- src (inverse)
    for each neighbor u of w
        if W[u] == 1
            continue
        alt = dist[w] + weight(w,u) // κόστος του src -> ... -> w
        if alt < dist[u] // καλύτερο από πριν, update
            dist[u] = alt
            prev[u] = w
```
Using a priority queue

• Finding the $w \notin W$ with minimum $\Delta[w]$ is slow
  - $O(n)$ time

• But we can use a priority queue for this!
  - We only keep vertices $w \notin W$ in the queue
  - They are compared based on their $\Delta[w]$
    (each $w$ has “priority” $\Delta[w]$)

• Careful when $\Delta[w]$ is modified!
  - Either use a priority queue that allows updates
  - Or insert multiple copies of each $w$ with different priorities
    ◦ the queue might contain already visited vertices: ignore them
Dijkstra's algorithm with priority queue

// Δεδομένα
src : αρχικός κόμβος
dest : τελικός κόμβος

// Πληροφορίες που κρατάμε για κάθε κόμβο u
W[u] : 1 αν o ν είναι στο σύνολο W, θ διαφορετικά
dist[u] : ο πίνακας Δ
prev[u] : o προηγούμενος του ν στο βέλτιστο μονοπάτι
pq : Priority queue, εισάγουμε tuples {u,dist[u]}
    συγκρίνοντας με βάση το dist[u]

// Αρχικοποίηση: W={} (εναλλακτικά μπορούμε και W={src})
prev[src] = NULL
dist[src] = 0
pq=queue_insert(pq, {src,0}) // dist[src] = 0
Dijkstra's algorithm with priority queue

```cpp
// Κύριως αλγόριθμος
while pq is not empty
    w = pqueue_max(pq)  // w with minimal dist[u]
pqueue_remove_max(pq)

    if exists(W[w])  // το w μπορεί να υπάρξει πολλές φορές στην o
        continue  // δεν κάνουμε replace), και να είναι ήδη vis
    W[w] = 1
    if w == dest
        stop  // optimal cost/path same as before

    for each neighbor u of w
        if exists(W[u])
            continue
        alt = dist[w] + weight(w,u)  // cost of src->...->w->u
        if !exists(dist[u]) OR alt < dist[u]
            dist[u] = alt
            prev[u] = w
            pqueue_insert(pq, {u,alt})  // προαιρετικά: replace αν υπ
    stop  // pq άδειασε πριν βρούμε το dest => δεν υπάρχει μονοπάτι
```
Notation

- \( s \rightarrow d \)
  - Direct step step from \( s \) to \( d \)

- \( s \overset{W}{\rightarrow} d \)
  - Multiple steps \( s \rightarrow \ldots \rightarrow d \)
  - All intermediate steps belong to the set \( W \subseteq V \)

- \( s \overset{W}{\longrightarrow} d \)
  - Shortest path among all \( s \overset{W}{\rightarrow} d \)
  - So \( s \overset{V}{\longrightarrow} d \) is the overall shortest one
Proof of correctness

• We need to prove that $\Delta[u]$ is the \textbf{minimum distance to } $u$
  - \textbf{after} the algorithm finishes

• But it's much easier to reason \textbf{step by step}
  - we need a property that holds \textbf{at every step}
  - this is called an \textbf{invariant} (property that never changes)
Proof of correctness

Invariant of Dijkstra's algorithm

- $\Delta[u]$ is the cost of the shortest path passing only through $W$
- And the shortest overall when $u \in W$

Formally:

1. For all $u \in V$ the path $s \xrightarrow{W} u$ has cost $\Delta[u]$
2. For all $u \in W$ the path $s \xrightarrow{V} u$ has cost $\Delta[u]$

Proof: induction on the size of $W$, for both (1), (2) together
Proof of correctness

Base case $W = \{s\}$

- Trivial, the only path $s \xrightarrow{W} u$ is the direct one $s \rightarrow u$
- For (1): its cost is exactly $T[s, u] = \Delta[u]$
  - initial value of $\Delta[u]$
- For (2): the only $u \in W$ is $s$ itself
Proof of correctness

Inductive case

• Assume $|W| = k$ and (1), (2) hold

• The algorithm
  - Updates $W$, adding a new vertex $w \notin W$
  - Updates $\Delta[u]$ for all neighbours $u$ of $w$

• Let $W'$, $\Delta'$ be the values after the update

• Show that (1), (2) still hold for $W'$, $\Delta'$
Proof of correctness

We start showing that (2) still holds for $W', \Delta'$

- The interesting vertex is the $w$ we just added
  - Vertices $u \neq w$ are trivial from the induction hypothesis

- Show: $s \xrightarrow{V} w$ has cost $\Delta'[w]$
  - Note: $\Delta'[w] = \Delta[w]$ (we do not update $\Delta[w]$)
  - We already know that $s \xrightarrow{W} w$ has cost $\Delta[w]$ (ind. hyp)
  - So we just need to prove that there is no better path outside $W$
Proof of correctness

• Assuming such path exists, let \( r \) be its **first** vertex outside \( W \)
  
  - So the path \( s \xrightarrow{W} r \xrightarrow{V} w \) has cost \( c < \Delta[w] \)
  
  - So the path \( s \xrightarrow{W} r \) has cost at most \( c < \Delta[w] \) (no negative weights!)
  
  - So \( \Delta[r] < \Delta[w] \)

• **Impossible!** We chose \( w \) to be the one with min \( \Delta[w] \)
Proof of correctness

It remains to show (1) for \( \mathcal{W}', \Delta' \)

• Take some arbitrary \( u \)
  - Let \( c \) be the cost of \( s \xrightarrow{\mathcal{W}'} u \)
  - Show: \( c = \Delta'[u] \)

• Three cases for the optimal path \( s \xrightarrow{\mathcal{W}'} u \)

• Case 1: the path does not pass through \( w \)
  - So it is of the form \( s \xrightarrow{\mathcal{W}} u \)
  - This path has cost \( \Delta[u] \) (induction hypothesis)
  - No update: \( \Delta'[u] = \Delta[u] = c \)
Proof of correctness

- Case 2: \( w \) is right before \( u \)
  - So the path is of the form \( s \overset{W}{\rightarrow} w \rightarrow u \)
  - The cost of \( s \overset{W}{\rightarrow} w \) is \( \Delta[w] \) (induction hypothesis)
  - So \( c = \Delta[w] + T[w, u] \)
  - So the algorithm will set \( \Delta'[u] = \Delta[w] + T[w, u] \) when updating the neighbours of \( w \)
  - So \( c = \Delta'[u] \)
Proof of correctness

- Case 3: some other \( x \) appears after \( w \) in the path
  - So the path is of the form \( s \xrightarrow{W} w \rightarrow x \xrightarrow{W} u \)
  - But the path \( s \xrightarrow{W} w \rightarrow x \) is no shorter than \( s \xrightarrow{W} x \)
    - From the induction hypothesis for \( x \in W \)
  - So \( s \xrightarrow{W} x \rightarrow u \) is also optimal, reducing to case 1!
Complexity

Without a priority queue:

• Initialization stage: loop over vertices: $O(n)$

• The while-loop adds one vertex every time: $n$ iterations

• Finding the new vertex loops over vertices: $O(n)$
  - same for updating the neighbours

• So total $O(n^2)$ time
Complexity

With a priority queue:

- Initialization stage: loop over vertices, so $O(n)$

- Count the number of updates (steps in the inner loop)
  - Once for every neighbour of every node: $e$ total
  - Each update is $O(\log n)$ (at most $n$ elements in the queue)

- Total $O(e \log n)$
  - Assuming a connected graph ($e \geq n$)
  - And an implementation using adjacency lists

- Only good for sparse graphs!
  - But $O(n \log n)$ can be hugely better than $O(n^2)$
The all-pairs shortest path problem

• Find the shortest path between all pairs \( s, d \)

• **Floyd-Warshall** algorithm

• Any weights
  - Even negative
  - But no **negative loops** (why?)
The all-pairs shortest path problem

Main idea

• At each step we compute the shortest path through a \textit{subset of vertices}
  - Similarly to $W$ in Dijkstra
  - But now the set at step $k$ is $W_k = \{1, \ldots, k\}$
    ◦ Vertices are numbered in any order

• Step $k$: the cost of $i \xrightarrow{W_k} j$ is $A_k[i, j]$
  - Similar to $\Delta$ in Dijkstra (but for all \textit{pairs} $i, j$ of nodes)
Floyd-Warshall algorithm

• The algorithm at each step computes $A_k$ from $A_{k-1}$

• Initial step $k = 0$
  - Start with $A_0[i, j] = T[i, j]$
  - Only direct paths are allowed
Floyd-Warshall algorithm

$k$-th iteration: the optimal $i \xrightarrow{W_k} j$ either passes thorough $k$ or not.

$$A_k[i, j] = \min \left\{ A_{k-1}[i, j], A_{k-1}[i, k] + A_{k-1}[k, j] \right\}$$
Floyd-Warshall algorithm in pseudocode

```c
void floyd_warshall() {

    for (int i = 0; i <= size-1; i++)
        for (int j = 0; j <= size-1; j++)
            A[i][j] = weight(i, j)

    for (int i = 0; i <= size-1; i++)
        A[i][i] = 0;

    for (int k = 0; k <= size-1; k++)
        // Compute A_k from A_{k-1}
        for (int i = 0; i <= size-1; i++)
            for (int j = 0; j <= size-1; j++)
}
```

A is the current $A_k$ at every step $k$. 


Complexity

• Three simple loops of $n$ steps

• So $O(n^3)$

• **Not** better than $n$ executions of Dijkstra in complexity
  - But much simpler
  - Often faster in practice
  - And works for **negative** weights
Readings

• T. A. Standish. *Data Structures, Algorithms and Software Principles in C.* Chapter 10

• A. V. Aho, J. E. Hopcroft and J. D. Ullman. *Data Structures and Algorithms.* Chapters 6 and 7